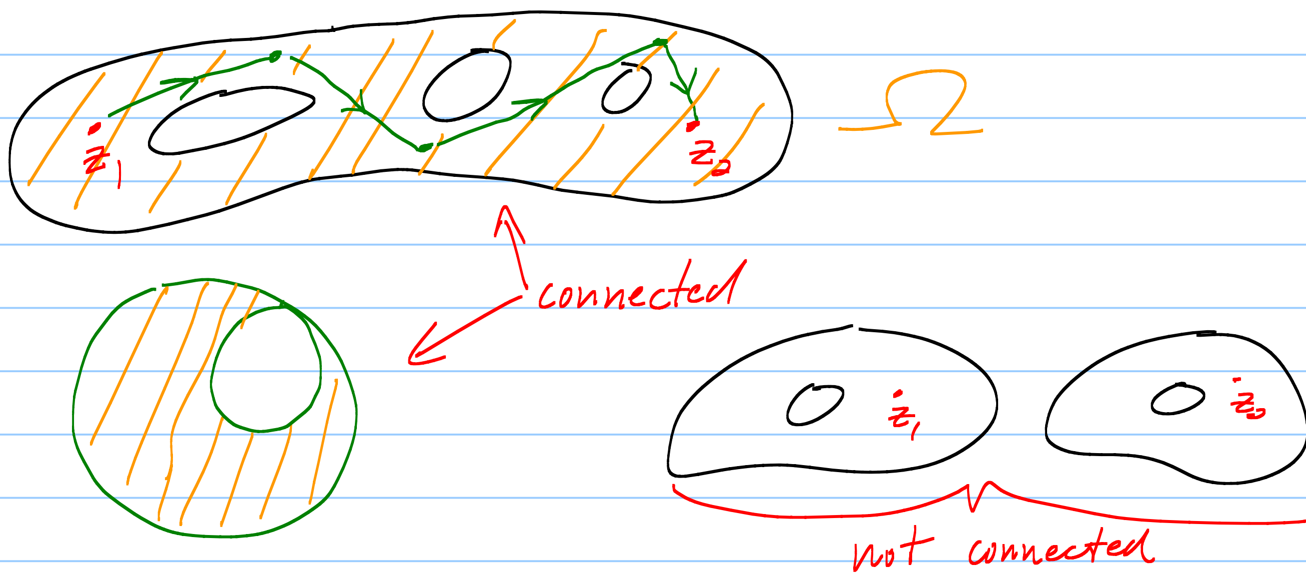
Lesson 3 Topology of $\mathbb{C}$ and analytic functions

Def^n: $\Omega \subset \mathbb{C}$ is *open* if, for each $z_0 \in \Omega$, there is a disc $D_R(z_0) \subset \Omega$.

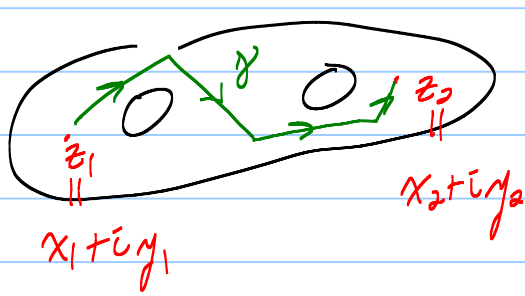
Def^n: An open set $\Omega \subset \mathbb{C}$ is *connected* if, given any two points $z_1, z_2 \in \Omega$, there is a polygonal path joining z_1 to z_2 *in* Ω.



Def^n: A *domain* is an open connected set in $\mathbb{C}$.

Lemma: Suppose $u(x,y)$ is a continuously diff'ble fcn on a domain Ω, and $\nabla u \equiv 0$, then $u \equiv c$, a constant on Ω.

Pf:



$$u(x_2, y_2) - \overbrace{u(x_1, y_1)}^{C} = \int_\gamma \underbrace{\nabla u}_{\equiv 0} \cdot d\vec{s} = 0$$

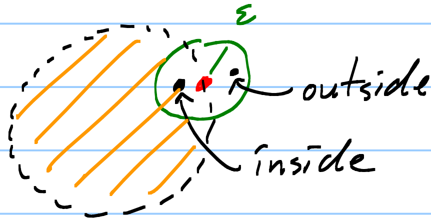
Hold z_1 fixed. Let z_2 slide. ✓

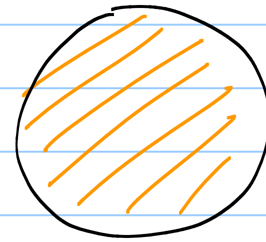
<u>Def</u>

Def^n: A point z_0 is a boundary point of a set $S \subset \mathbb{C}$ if, for every $\varepsilon > 0$ (no matter how small),

$D_\varepsilon(z_0) - \{z_0\}$ contains a point inside S and a point outside S.

EX:

open



closed

Def^n: A set $S \subset \mathbb{C}$ is closed if it contains all its boundary points.


not open
not closed

Calculus on $\mathbb{C}$:

Limits: Sequence $\{z_n\}$ in $\mathbb{C}$ converges to z_0 means, given $\varepsilon > 0$, there is an N such that $|z_n - z_0| < \varepsilon$ when $n > N$.

Write $\lim_{n \to \infty} z_n = z_0$.

Similarly $\lim_{z \to z_0} f(z) = L$ means, given $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ when $\underset{z \neq z_0}{|z - z_0| < \delta}$.

[Note: $f(z_0)$ does not need to defined.]

Def^n: f is continuous if $\lim_{z \to z_0} f(z) = f(z_0)$.

EX: $f(z) = z^2$

$$DQ = \frac{f(z)-f(a)}{z-a} = \frac{z^2-a^2}{z-a} = \frac{(z-a)(z+a)}{z-a} = z+a$$

not defined at $z=a$

$\to 2a$ as $z \to a$.

Why: Let $\varepsilon > 0$. $|DQ - 2a| = |(z+a) - 2a| = |z-a| < \varepsilon$

want

Aha! Take $\delta = \varepsilon$.

z^2 is "complex diff'ble"

Def^n: A function $f : \Omega \to \mathbb{C}$ mapping a domain Ω into $\mathbb{C}$ is called analytic on Ω if it is complex diff'ble at each point in Ω.

MA 425/525 is about analytic fcns.

Basic facts of real analysis hold $\mathbb{C} \to \mathbb{C}$:

EX: $\lim_{z \to z_0} f(z) = A$ $\quad \lim_{z \to z_0} g(z) = B$

Then $\lim_{z \to z_0} [f(z) + g(z)] = A + B$

$\lim_{z \to z_0} f(z) g(z) = AB$

$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$ provided that $B \neq 0$.

Consequence: Complex polys are continuous.

$$P(z) = a_N z^N + \cdots + a_1 z + a_0$$

Rational fcns $\frac{P(z)}{Q(z)}$ are cont. where $Q(z) \neq 0$.

Def^n: f is complex diff'ble at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ exists. If it does, call it $f'(z_0)$.

Fact: Analysis proofs $\mathbb{R} \to \mathbb{R}$ translate word for word into $\mathbb{C} \to \mathbb{C}$.

Because $|z+w| \le |z| + |w| \leftarrow$ true in $\mathbb{C}$

$$|z-w| \ge \big||z| - |w|\big|$$

plus same definitions.

Differentiation rules hold $\mathbb{C} \to \mathbb{C}$:

$$(f+g)' = f' + g'$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2} \quad \text{where } g(z) \neq 0.$$

Fact: Complex polynomials are analytic on $\mathbb{C}$.

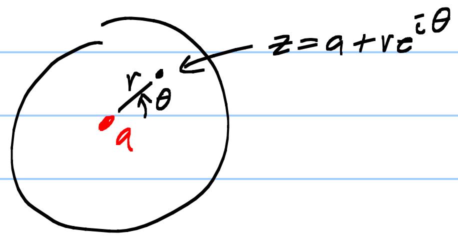
Ques: Is e^z complex diff'ble? Yes!

EX: $f(z) = \bar{z}$ is not! $\qquad f(x+iy) = x - iy$

$$\frac{\bar{z} - \bar{a}}{z - a} = \frac{\overline{(a + re^{i\theta})} - \bar{a}}{(a + re^{i\theta}) - a}$$

Write $z = a + re^{i\theta}$

$$= \frac{\overline{e^{i\theta}}}{e^{i\theta}} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-i\theta - i\theta} = e^{-2i\theta}$$



ouch! different for sliding into a along different θ's!