Lesson 6: Cauchy-Riemann eqns

Theorem: If u and v are C^1-smooth and satisfy the

C-R Eqns $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$, then

$f(x+iy) = u(x,y) + i\,v(x,y)$ is analytic.

Taylor's Thm with remainder: u C^1-smooth:

$$u(x,y) = \underbrace{u(x_0,y_0)}_{u^0} + \underbrace{u_x(x_0,y_0)}_{u_x^0}(x-x_0) + \underbrace{u_y(x_0,y_0)}_{u_y^0}(y-y_0) + \underbrace{R_u(x,y)}_{\lim\limits_{(x,y)\to(x_0,y_0)} \frac{R_u}{|z-z_0|} = 0}$$

$$v(x,y) = \cdots \qquad \cdots \qquad + R_v(x,y)$$

Pf of C-R Thm: $DQ = \dfrac{f(z)-f(z_0)}{z-z_0} = \dfrac{[u+iv]-[u^0+iv^0]}{z-z_0}$

$$= \frac{u_x^0(x-x_0) + \underbrace{u_y^0}_{=-v_x^0}(y-y_0) + i\left[v_x^0(x-x_0) + \underbrace{v_y^0}_{=u_x^0}(y-y_0)\right]}{z-z_0} + \underbrace{\frac{R_u + iR_v}{z-z_0}}_{\mathcal{R}}$$

Aha! Complex mult.

$$= \frac{[u_x^0 + i v_x^0]\left((x-x_0) + i(y-y_0)\right)}{z-z_0} + \mathcal{R}$$

$$= \boxed{u_x^0 + i v_x^0} + \mathcal{R}$$

where $|\mathcal{R}| \le \dfrac{R_u}{|z-z_0|} + \dfrac{R_v}{|z-z_0|} \longrightarrow 0$ as $z \to z_0$.

So $f'(z_0)$ exists and given by Red box #1 formula.

$\mathbb{R}^2 \to \mathbb{R}^2$ v.s. $\mathbb{C} \to \mathbb{C}$

$(x,y) \mapsto u(x,y) + i v(x,y)$	$z \mapsto f(z)$
Jacobian matrix:	
$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$	$f'(z)$ exists
$= \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$	
↑ C-R Eqns!	
Represents complex multiplication!	

Exciting consequence: Real and imaginary parts of analytic fcns are harmonic!

Thm: Suppose u and v are $\mathcal{C}^2$-smooth and satisfy the C-R Eqns (so $u+iv$ is analytic). Then u and v are harmonic $\left[\Delta u \equiv 0,\ \Delta v \equiv 0 \text{ where } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.\right]$

Why: C-R Eqns $\begin{cases} u_x = v_y & (A) \\ u_y = -v_x & (B) \end{cases}$

$\frac{\partial}{\partial x}(A)$: $u_{xx} = v_{xy}$

$+$ $\frac{\partial}{\partial y}(B)$: $u_{yy} = -v_{yx}$

Aha! $v_{xy} = v_{yx}$ when v is $\mathcal{C}^2$-smooth.

$$\Delta u = 0 \quad \checkmark$$

Similarly for v: $\frac{\partial}{\partial y}(A) - \frac{\partial}{\partial x}(B)$ etc ...

EX: $z^2 = (x^2 - y^2) + i\,2xy$ ✓

$e^z = e^x \cos y + i\, e^x \sin y$ ✓

Problem: Given C^2-smooth harmonic fcn u, can we find a C^2-smooth v such that $u+iv$ is analytic.

Yes! Here's how: Given u, want v satisfying

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad \begin{matrix} v_x = -u_y \\ v_y = u_x \end{matrix}$$

$$\underset{\substack{\uparrow \\ \text{potential} \\ \text{fcn!}}}{\nabla v} = \underbrace{-u_y\,\hat{i} + u_x\,\hat{j}}_{\vec{F}}$$

Aha! Can find v if $\vec{F}$ is conservative, which happens exactly when $\operatorname{Curl} \vec{F} = \vec{0}$

[Hmmm:

$$\operatorname{Curl} \vec{F} = \operatorname{Det} \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -u_y & u_x & 0 \end{bmatrix}$$

$$= (\underbrace{u_{xx} + u_{yy}}_{=0})\,\hat{k}$$

Yes! $\operatorname{Curl} \vec{F} = \vec{0}$]

So v exists!

Another way to remember the Curl $\vec{F} = 0$ condition:

If $\nabla v = \vec{F}$

$$v_x \hat{i} + v_y \hat{j} = F_1 \hat{i} + F_2 \hat{j}$$

$v_x = F_1$ (A) $\qquad \frac{\partial}{\partial y}$(A): $v_{yx} = \frac{\partial F_1}{\partial y}$

$v_y = F_2$ (B) $\qquad \frac{\partial}{\partial x}$(B): $v_{xy} = \frac{\partial F_2}{\partial x}$

If v exists, then mixed partials are equal : $\boxed{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}}$

Necessary Cond.

Math 261: Nec. Cond is also sufficient!

EX: $u(x,y) = x^2 - y^2 + xy + 3 \qquad \Delta u \equiv 0$ ✓

Want v with
$$\begin{cases} \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y - x & \text{(A)} \\ \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x + y & \text{(B)} \end{cases}$$

Use (A): $\quad v = \int 2y - x \, dx$ ← "Partial integration" Treat y like a constant

$$= \underline{2xy - \tfrac{1}{2}x^2 + C(y)}$$ ← where C might depend on y.

Use (B): $\quad \frac{\partial}{\partial y}\underbrace{\left[2xy - \tfrac{1}{2}x^2 + C(y)\right]}_{v} \overset{\text{want}}{=} 2x + y$

$$2x + C'(y) \overset{\text{want}}{=} 2x + y$$

Miracle! All x's cancel! $\boxed{C'(y) = y}$

So $C(y) = \int y\,dy = \frac{1}{2}y^2 + k$

Done! $\underline{v = 2xy - \frac{1}{2}x^2 + \frac{1}{2}y^2 + k}$ See that v is unique up to $+k$.

Hmmm, $u + iv = f(z)$ $z = x + iy$ $\bar{z} = x - iy$ $x = \frac{z + \bar{z}}{2}$ $y = \frac{z - \bar{z}}{2i}$

$u + iv =$ Mess in terms of z and $\bar{z}$ $= \cdots$ all $\bar{z}$'s cancel! get $f(z)$.