Lesson 8 Complex exponential

Lessons 7, 8 due Wed

Look for an analytic fcn $E(z)$ such that $E'(z) = E(z)$ and $E(0) = 1$.

Write $E(x+iy) = u(x,y) + i v(x,y)$

Want $E'(z) = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} \overset{\text{want}}{=} \underbrace{u + iv}_{E(z)}$ (A) (B)

(B): $k'(y)e^x - c'(y)e^x = c(y)e^x + i\, k(y)e^x$

(A): Need $\underbrace{u_x = u}$, $\underbrace{v_x = v}$

$u = c(y)e^x \qquad v = k(y)e^x$

(B): $\begin{cases} k'(y) = c(y) & (1) \\ -c'(y) = k(y) & (2) \end{cases}$

(1)': $\underbrace{k'' = c'}$

plug into 2:

Get $k'' = -k$.

Play around: Get $\begin{cases} k'' + k = 0 \\ c'' + c = 0 \end{cases}$

So $\begin{cases} k(y) = c_1 \cos y + c_2 \sin y \\ c(y) = k_1 \cos y + k_2 \sin y \end{cases}$

Finally, $E(0) = 1$ pins them down.

Get $E(x+iy) = e^x \cos y + i e^x \sin y$ ✓

Properties:

1) $E(0) = 1$
2) $E'(z) = E(z)$
3) $E(z) \neq 0$ for all z
4) $E(z+w) = E(z)E(w)$

5) $E(-z) = 1/E(z)$. So $E(z-w) = E(z)/E(w)$

6) $E(z) = 1 \iff z = 2\pi i n$, $n \in \mathbb{Z}$.

Lemma: If f is analytic on $\mathbb{C}$ (or $D_R(0)$) and $f'(z) = k f(z)$,

then $f(z) = c\,E(kz)$ where $c = f(0)$.

Why: Assume $f' - kf = 0$ ← MA 262: Multiply by integrating factor $u = E(-kz)$

$$E(-kz)\, f'(z) \underbrace{-\, k E(-kz)}\, f(z) = 0$$

Aha! $\frac{d}{dz} E(-kz) = \underbrace{E'(-kz)}_{E(-kz)}\frac{d}{dz}(-kz)$

$E(-kz)\cdot(-k)$ ✓

Product rule: $\frac{d}{dz}\Big[\underbrace{E(-kz)\, f(z)}_{\text{Must be constant } c}\Big] \equiv 0$

So $E(-kz) f(z) = c$. Plug in $z = 0$ too see $c = f(0)$.

Conclude $f(z) = \dfrac{c}{E(-kz)}$.

Hmmm. Notice that $f(z) = E(kz)$ satisfies $f' = kf$.

Do the above: Get $E(kz) = \dfrac{1}{E(-kz)}$. Done!

Finally $f(z) = f(0)\, E(kz)$. ✓

Very cool thing: $f(z) = E(z+w)$ $\boxed{f(0) = E(w)}$ ← C

Then $\quad f'(z) = E'(z+w) \cdot \frac{d}{dz}[z+w]$

$$= E(z+w) \cdot 1 = \underset{\nwarrow k=1}{f(z)}$$

Lemma $\Rightarrow \quad \underbrace{f(z)}_{E(z+w)} = c\, E(1 \cdot z) = E(w)\, E(z)$

4) is proved!

$\underbrace{E(z-z)}_{E(0)=1} = E(z) \cdot E(-z)$ $\qquad$ So $E(z)$ is never zero.

And $\quad E(-z) = 1/E(z)$.

Last one: $\quad E(x+iy) = \underbrace{e^x \cos y}_{1} + i\, \underbrace{e^x \sin y}_{0} = 1 + i\,0$

$\sin y = 0 \Rightarrow \boxed{y = n\pi}$, $n \in \mathbb{Z}$.

But $\quad \cos n\pi = \begin{cases} -1 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$

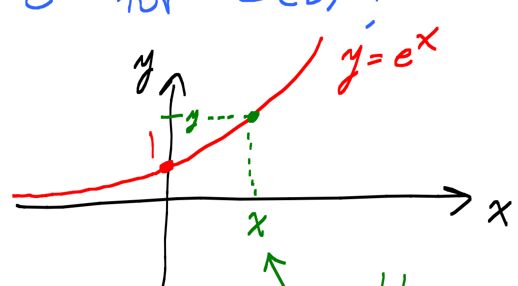
To have $e^x \cos y = 1$, must have n even.

$e^x \cdot 1 = 1 \leftarrow$ must have $x = 0$.

So $\quad z = 0 + 2\pi n i \quad \checkmark$

From now on, write e^z for $E(z)$!

Real log fcn: $\quad \ln x$

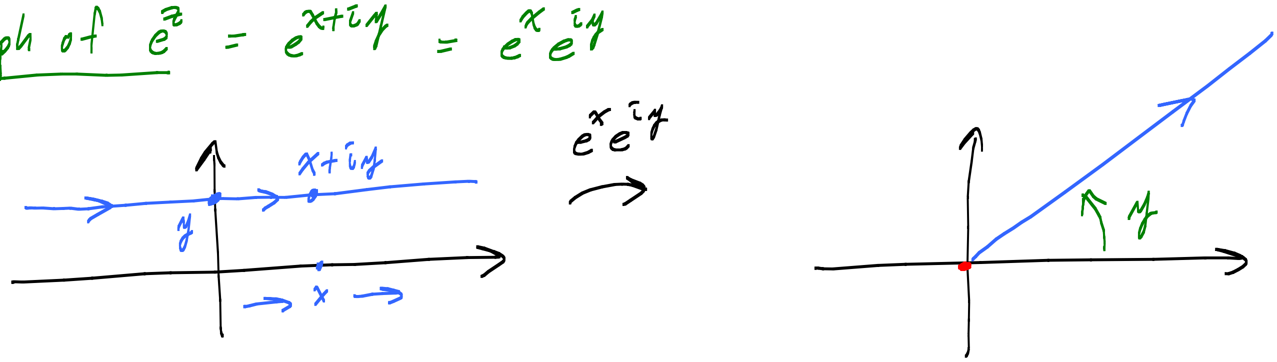


$\nwarrow$ exactly one $\quad x = \ln y$

$= \text{inverse of } \exp$

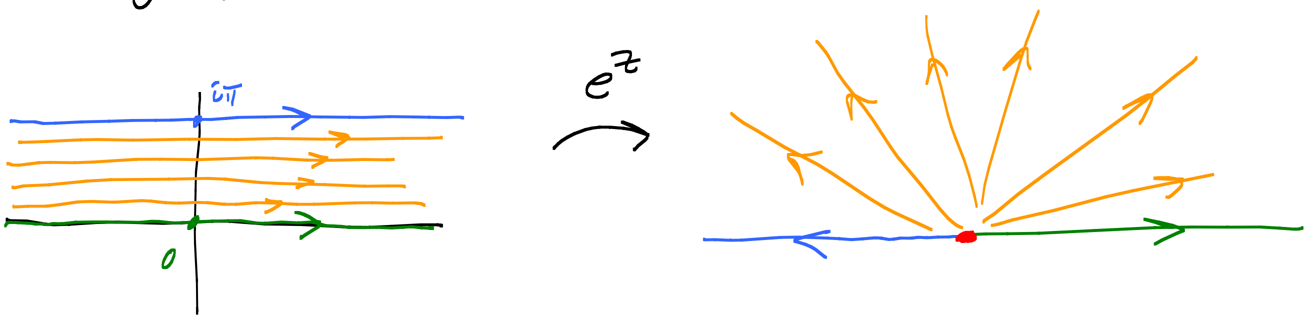
$\operatorname{Ln} x$ only defined for $x > 0$.

Graph of $e^z = e^{x+iy} = e^x e^{iy}$

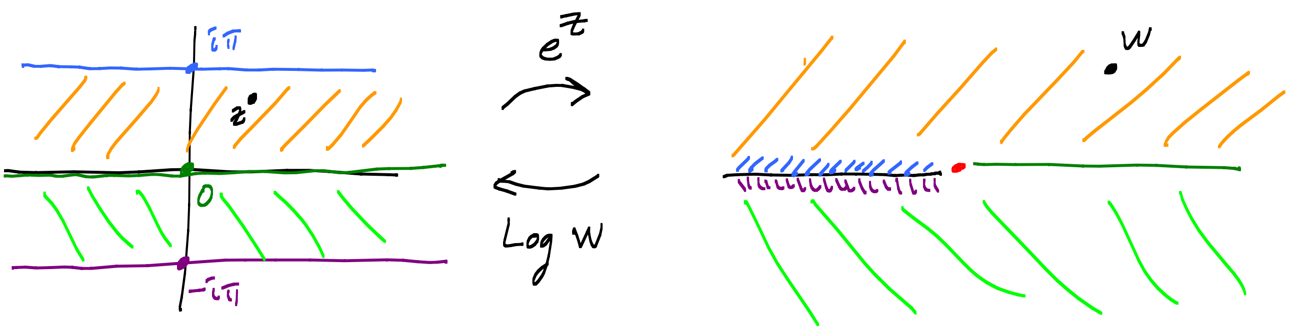


$-\infty < x < \infty$

$0 < e^x < \infty$



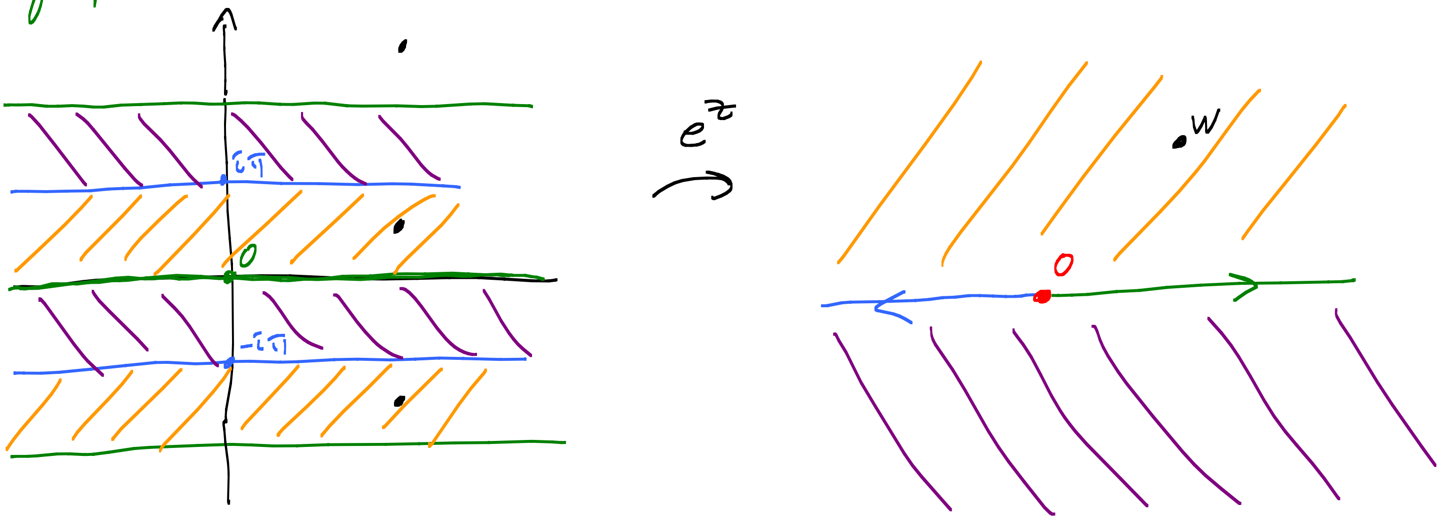
e^z maps strip $\{z : 0 < \operatorname{Im} z < \pi\}$ one-to-one onto the Upper Half Plane $\{z : \operatorname{Im} z > 0\} \leftarrow$ UHP



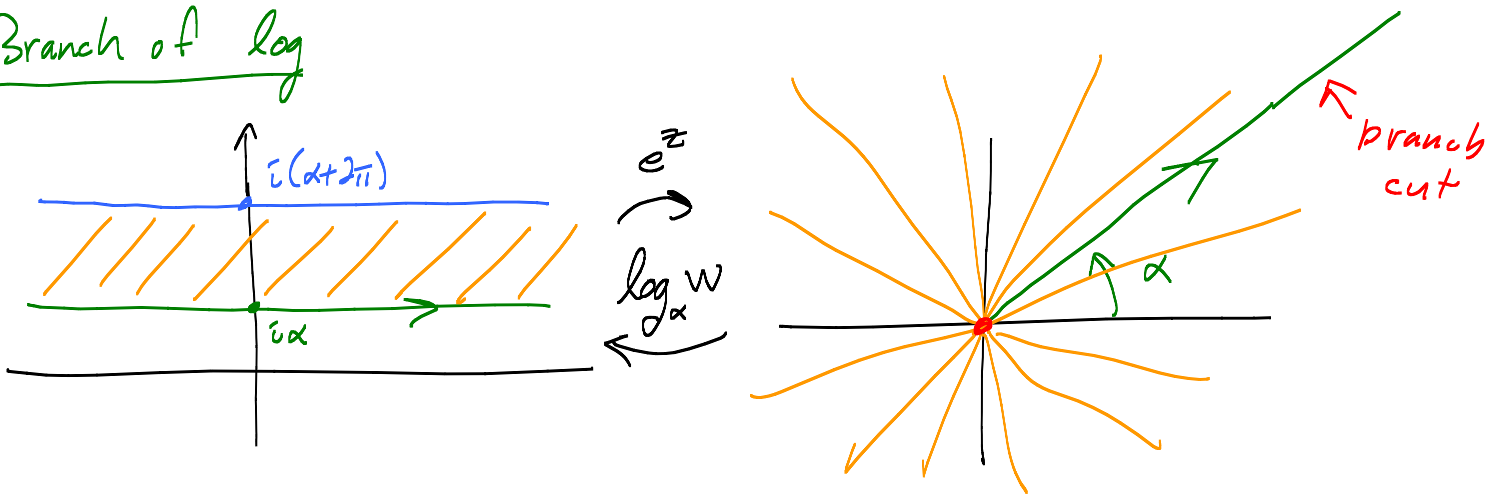
$\operatorname{Log} w = \operatorname{Ln}|w| + i \operatorname{Arg} w$

$$e^z : \{z : -\pi < \operatorname{Im} z < \pi\} \xrightarrow[\text{onto}]{1-1} \mathbb{C} - (-\infty, 0]$$

Big picture



Branch of log



$$e^z : \{z : \alpha < \operatorname{Im} z < \alpha + 2\pi\} \xrightarrow[\text{onto}]{1-1} \mathbb{C} - \{re^{i\alpha} : r \geq 0\}$$

$\log_\alpha w$ is the inverse.