Lesson 12 Complex integrals

Two kinds of fuctions: $z(t): \mathbb{R} \to \mathbb{C}$ $\quad z(t) = x(t) + iy(t)$

$f(z): \mathbb{C} \to \mathbb{C}$ $\quad f(x+iy) = u(x,y) + iv(x,y)$

Two kinds of derivatives: $z'(t) = x'(t) + iy'(t)$

$$f'(z) = \lim DQ = \begin{cases} u_x + iv_x \\ v_y - iu_y \end{cases}$$ C-R Eqns

Two kinds of integrals:

$$\int_a^b z(t)\,dt = \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$$

$\gamma: z(t),\ a \le t \le b$

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,\underbrace{z'(t)\,dt}_{dz}$$

Two kinds of chain rules:

Chain rule #1: $h(z) = f(g(z))$ where f, g analytic, then

$$h'(z) = f'(g(z))\,g'(z)$$

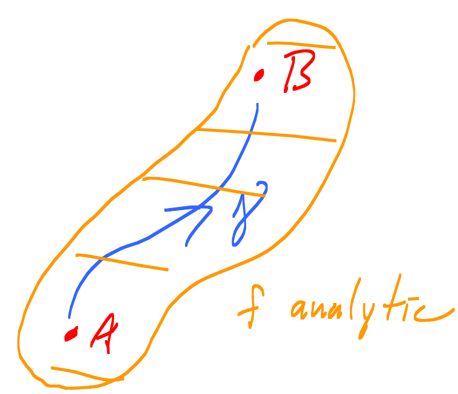
Chain rule #2: f analytic and $z(t) = x(t) + iy(t)$,

then $$\frac{d}{dt} f(z(t)) = f'(z(t))\,z'(t)$$

Two Fund. Theorems of Calculus:

1) $$\int_a^b z'(t)\,dt = z(b) - z(a)$$

2) $$\int_\gamma f'(z)\,dz = f(B) - f(A)$$



Two Basic Estimates:

1) $$\left| \int_a^b z(t)\,dt \right| \le \int_a^b |z(t)|\,dt$$

2) $$\left| \int_\gamma f(z)\,dz \right| \le \underbrace{\left(\max_{z \in tr(\gamma)} |f(z)| \right)}_{M} \text{Length}(\gamma)$$

$$tr(\gamma) = \text{"trace of } \gamma\text{"} = \{ z(t) : a \le t \le b \}$$

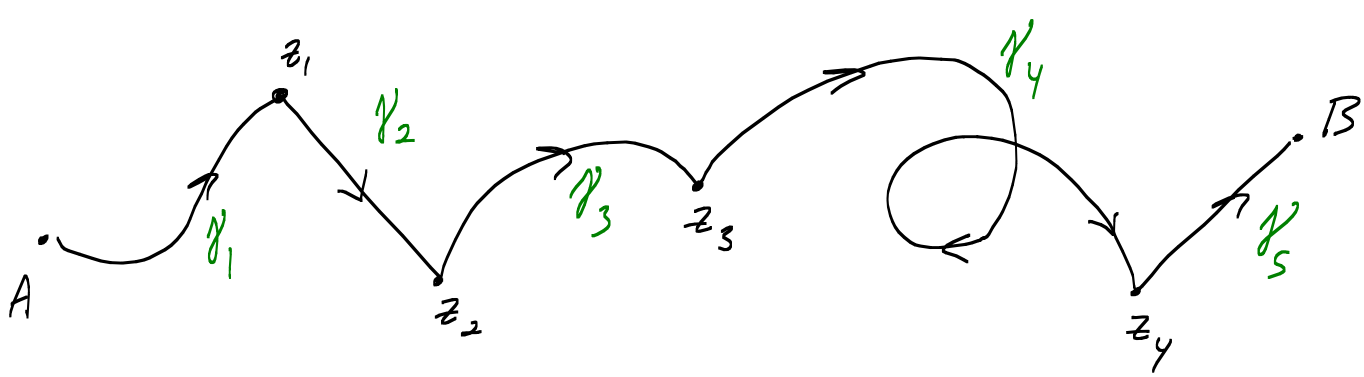
$$M = \text{Max} \{ |f(z(t))| : a \le t \le b \}$$

Max of a cont. fcn. on a closed interval is attained and finite

Why: $$\left| \int_\gamma f\,dz \right| = \left| \int_a^b f(z(t))\,z'(t)\,dt \right|$$

$$\underset{\text{Basic est. \#1}}{\le} \int_a^b \underbrace{|f(z(t))|}_{\le M} |z'(t)|\,dt \le M \underbrace{\int_a^b \underbrace{|z'(t)|}_{\sqrt{x'(t)^2 + y'(t)^2}}\,dt}_{= \text{length}(\gamma)} \checkmark$$

Important fact: I've been assuming γ is C^1-smooth, but all our facts above hold for "piecewise C^1-smooth" curves.



<u>Why</u>: $\int_\gamma f' dz = \sum \int_{\gamma_k} f' dz$

cancel

$= \left[f(z_1) - \underline{f(A)}\right] + \left[f(z_2) - f(z_1)\right] + \cdots + \left[\underline{f(B)} - f(z_N)\right]$

$= f(B) - f(A)$ ✓ because of pairwise cancellation.

Coming soon: <u>Cauchy Theorem</u>: f analytic on a domain Ω with no holes. Then $\int_\gamma f\, dz = 0$ for any closed curve in Ω. Our way: 4.4 b using Green's Thm.

Funny thing: Reverse is true! <u>Must</u> study $\int_\gamma f\, dz$.

<u>Big fact</u>: $f(z)$ analytic on a domain Ω.

$f(z)$ has an analytic <u>anti-derivative</u> $\Longleftrightarrow \int_\gamma f\, dz = 0$ for <u>every</u> closed curve in Ω.

<u>Why</u>: ($\Rightarrow$) easy direction. If $f(z) = F'(z)$ on Ω,

then $\int_\gamma f\, dz = \int_\gamma F'\, dz \underset{\uparrow}{=} F(\text{END}) - F(\text{START}) = 0$

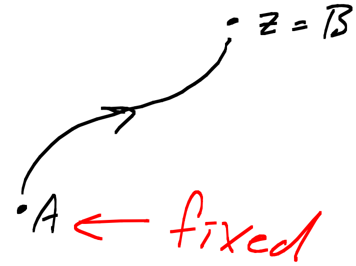
Fund. Thm. Calc #2

same when γ closed

($\Leftarrow$): Assume $\int_\gamma f\, dz = 0$ for closed γ in Ω.

Hmmm. If I had an F with $F' = f$:

$$\int_{\gamma_A^B} f\,dz = \int_{\gamma_A^B} F'\,dz = F(B) - \underbrace{F(A)}_{\text{const}}$$

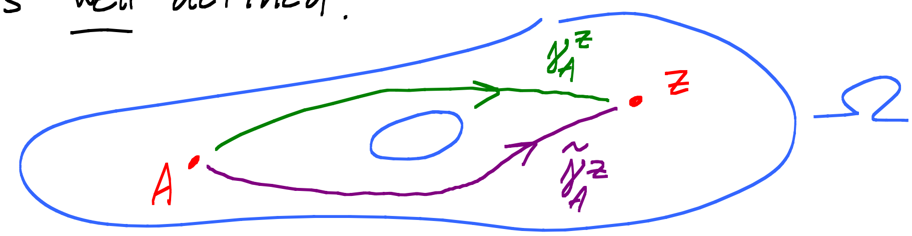


Aha! See what F needs to be!

"Define" $F(z) = \int_{\gamma_A^z} f(w)\,dw$

A fixed. γ starts at A, goes to z in Ω.

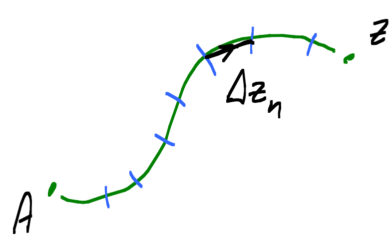
Claim: F is well defined.



Aha! $\gamma = \gamma_A^z \cup (-\tilde{\gamma}_A^z)$ is a closed curve.

So $$0 = \int_\gamma f\,dz = \int_{\gamma_A^z} f\,dz + \left(-\int_{\tilde{\gamma}_A^z} f\,dz\right) \checkmark$$

Why is $\int_{-\gamma} = -\int_\gamma$?



$\int_\gamma = \lim \sum f(z_n)\,\Delta z_n$

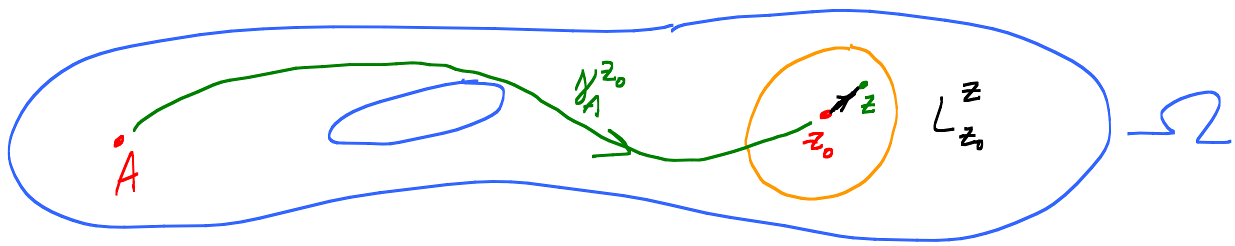
Δz's going backwards are $-\Delta z$'s going forwards

So $F(z) = \tilde{F}(z) \leftarrow$ using $\tilde{\gamma}_A^z$.

Step 2: Show $F'(z) = f(z)$.

Lemma: $\int_{\gamma_a^z} c\,dw = c(z-a)$ because $c = \frac{d}{dw}[cw]$.

Pf: $\displaystyle\int_{\gamma_a^z} c\,dw = \int_{\gamma_a^z} \frac{d}{dw}\left[cw\right] dw = cw\Big|_a^z = c(z-a)$ ✓



Aha! $\displaystyle DQ = \frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\int_{L_{z_0}^z} f\,dw \xrightarrow{\text{Claim}} f(z_0)$

f analytic at $z_0 \Longrightarrow f$ continuous at z_0.

So $\quad f(z) = f(z_0) + E(z) \quad$ where

$E(z) \to 0$ as $z \to z_0$.

$$DQ = \frac{1}{z-z_0}\Bigg[\underbrace{\int_{L_{z_0}^z} \underbrace{f(z_0)}_{c}\,dw}_{\substack{= f(z_0)(z-z_0) \\ \text{Lemma}}} + \int_{L_{z_0}^z} E(w)\,dw\Bigg]$$

$$= f(z_0) + \mathcal{E}(z) \qquad \text{where}$$

$$|\mathcal{E}(z)| = \frac{1}{|z-z_0|}\underbrace{\left|\int_{L_{z_0}^z} E(w)\,dw\right|}_{\le \Big(\underbrace{\max\limits_{L_{z_0}^z} |E(w)|}_{\substack{\to 0 \text{ as} \\ z \to z_0}}\Big)\cdot \underbrace{\text{Length}(L_{z_0}^z)}_{= |z-z_0|}} \longrightarrow 0 \ ✓$$

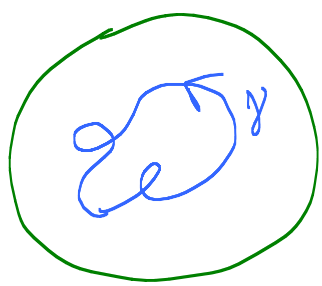
Remarks: Find an anti-derivative for z^N:

$$\frac{d}{dz}\left[\frac{1}{N+1} z^{N+1}\right]. \qquad \text{And} \quad \frac{d}{dz}[z] = 1$$

So complex polys have complex antiderivatives.

Big fact $\Rightarrow$ $\displaystyle\int_\gamma P(z)\,dz = 0$ P poly, γ closed in $\mathbb{C}$

Hmmm:



$$f = \lim_{n\to\infty} P_n(z)$$

$$\int_\gamma f\,dz = \lim_{n\to\infty} \underbrace{\int_\gamma P_n\,dz}_{0} = 0$$

So $\displaystyle\int_\gamma f\,dz$ when $f = \lim$ polys $\leftarrow$ think "power series"