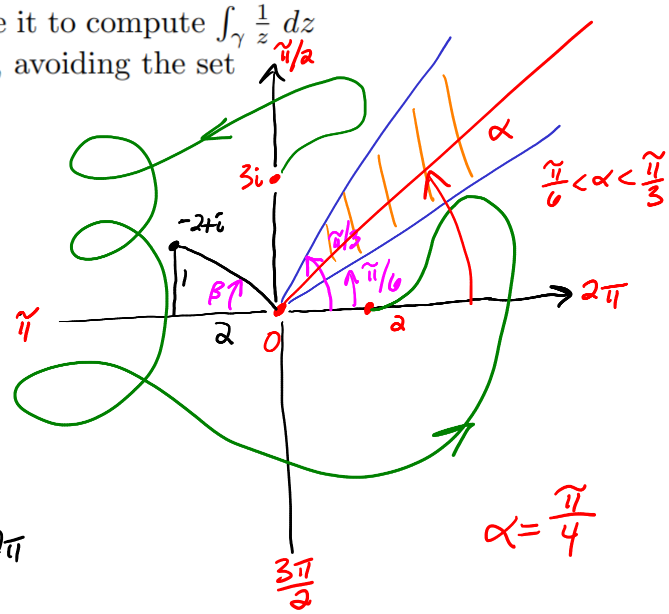
Review for Midterm Midterm exam in class Wed, March 5

(20) **3.** **a)** Define a branch of a complex log function and use it to compute $\int_\gamma \frac{1}{z}\,dz$ where γ is any curve that starts at $3i$ and ends at 2, avoiding the set

$$\{z = re^{i\theta} : r \geq 0,\ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}\}.$$

b) Compute $\int_\gamma \frac{1}{z^2}\,dz$. Explain your reasoning.



$$\log_\alpha z = \operatorname{Ln}|z| + i\theta$$

where $\theta \in \arg z$ with $\alpha < \theta < \alpha + 2\pi$

$$\frac{\pi}{4} < \theta < \frac{9\pi}{4}$$

$$\int_\gamma \frac{1}{z}\,dz = \int_\gamma \frac{d}{dz}\left[\log_\alpha z\right] dz = \log_\alpha z \Big|_{3i}^{2}$$

$$= \left(\operatorname{Ln} 2 + i\,2\pi\right) - \left(\operatorname{Ln} 3 + i\,\frac{\pi}{2}\right)$$

$$= \operatorname{Ln}\frac{2}{3} + i\,\frac{3\pi}{2}$$

$$\log_\alpha(-2+i) = \operatorname{Ln}\sqrt{(-2)^2+1^2} + i\left(\pi - \underbrace{\operatorname{Tan}^{-1}\frac{1}{2}}_{\beta}\right)$$

$$= \frac{1}{2}\operatorname{Ln} 5 + \ldots$$

$$\int_\gamma \frac{1}{z^2}\,dz = \int_\gamma \frac{d}{dz}\left[-\frac{1}{z}\right] dz = -\frac{1}{z}\Big|_{3i}^{2} = -\frac{1}{2} - \left(-\frac{1}{3i}\right)$$

$$= -\frac{1}{2} - \frac{1}{3}i$$

(20) **4.** Let C_R denote the half circle parametrized by $z(t) = Re^{it}$, $0 \le t \le \pi$. Use careful estimates to show that

$$\left| \int_{C_R} \frac{z-2}{z^7+5}\, dz \right|$$

tends to zero as $R \to \infty$.

$$\left| \frac{a+b}{c+d} \right| \le \frac{|a|+|b|}{\big||c|-|d|\big|}$$

$$\le \underbrace{\left(\max_{C_R} \left| \frac{z-2}{z^7+5} \right| \right)}\ \underbrace{(\pi R)}_{\text{Length}(C_R)}$$

$$\left| \frac{z-2}{z^7+5} \right| \le \frac{|z|+|2|}{\big||z^7|-|5|\big|} = \frac{R+2}{\underbrace{|R^7-5|}}$$

$|z| = R.$

$R > 5^{1/7}$, then $\leftarrow$ or $R > 2$

$|R^7 - 5| = R^7 - 5$

$$I \le \frac{R+2}{R^7-5} \cdot \pi R \quad \text{if} \quad R > 1$$

$$\longrightarrow 0 \quad \text{as} \quad R \to \infty.$$

$$\int_0^{\pi} \frac{Re^{it}-2}{R^7 e^{i7t}+5}\, iRe^{it}\, dt \qquad \text{Ugh!}$$

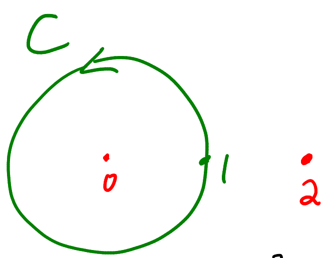
$\pi R = (\text{length of interval}) \cdot (\text{this } R)$

6. Let C denote the unit circle parameterized in the counterclockwise sense.
Compute $\displaystyle\int_C \frac{e^{3z}}{\underbrace{(2z-1)}_{[2(z-\frac{1}{2})]^2}{}^2(z-2)}\,dz$. Explain.

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{\underbrace{n+1}_{2}}}\,dz \qquad n=1$$

C ↖ counterclockwise!

[If clockwise, $= -\int_C \cdots$]

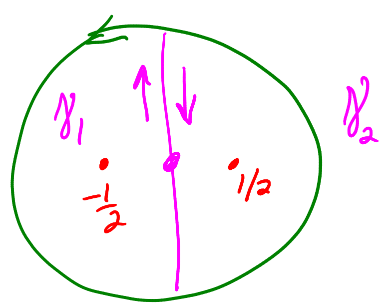


$$I = \int_C \frac{\left[\frac{1}{4}\frac{e^{3z}}{z-2}\right]}{(z-\frac{1}{2})^2}\,dz \quad \leftarrow f(z) \qquad = \frac{2\pi i}{1!}\left[\frac{1}{4}\,\frac{e^{3z}}{z-2}\right]'\Bigg|_{z=\frac{1}{2}}$$

What about $\displaystyle\int_{C_1} \frac{e^{3z}}{(z-1/2)^2(z+\frac{1}{2})}\,dz$

Partial fractions: $\displaystyle\frac{1}{(z-1/2)^2(z+\frac{1}{2})} = \frac{A}{(z-\frac{1}{2})^2} + \frac{B}{z-\frac{1}{2}} + \frac{C}{z+\frac{1}{2}}$

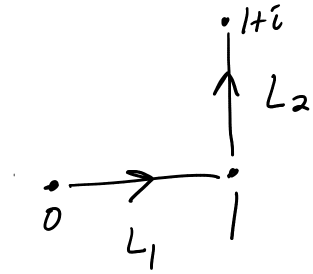
or

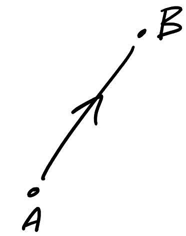


$$\int_C = \int_{\gamma_1} + \int_{\gamma_2}$$

10. Compute the following path integrals

a) $\int_\gamma |z|^2\,dz$ where γ is the line from 0 to 1 followed by the line from 1 to $1+i$.





$L_1 : t, \quad 0 \le t \le 1$

$L_2 : z(t) = 1 + [(1+i) - 1]t$

$= 1 + it, \quad 0 \le t\ 1$

$A + t(B-A)$

$0 \le t \le 1$

$$I = \int_{L_1} + \int_{L_2} = \int_0^1 t^2 \cdot \underbrace{1}_{z'(t)} \cdot dt + \int_0^1 |1+it|^2 \underbrace{(i)}_{z'(t)}\, dt$$

9. Compute $\int_0^\pi e^{3it}\,dt$ where t is a real variable.

$$\int_0^\pi [\cos 3t + i \sin 3t]\,dt = \int_0^\pi \cos 3t\,dt + i \int_0^\pi \sin 3t\,dt$$

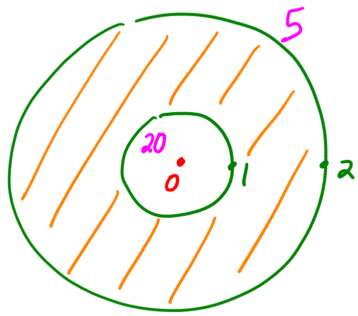
or

$$= \int_0^\pi \frac{d}{dt}\left[\frac{1}{3i} e^{3it}\right] dt = \frac{1}{3i} e^{3it}\Big|_0^\pi$$

3. Find a continuous real valued function u on the annulus $\{z : 1 \leq |z| \leq 2\}$ that is harmonic inside the annulus, equal to 20 on the inner boundary and equal to 5 on the outer boundary.

Key fact

$\operatorname{Ln}|z|$ is harmonic



Try $\quad A \operatorname{Ln}|z| + B$

on C_1: Need $\begin{cases} A \operatorname{Ln} 1 + B = 20 \\ A \operatorname{Ln} 2 + B = 5 \end{cases}$

on C_2: Need

Determine A, B

2. Show that $u(x, y) = xe^x \cos y - ye^x \sin y$ is harmonic on $\mathbb{C}$ and find a harmonic conjugate for u on $\mathbb{C}$.

Key C-R eqns.

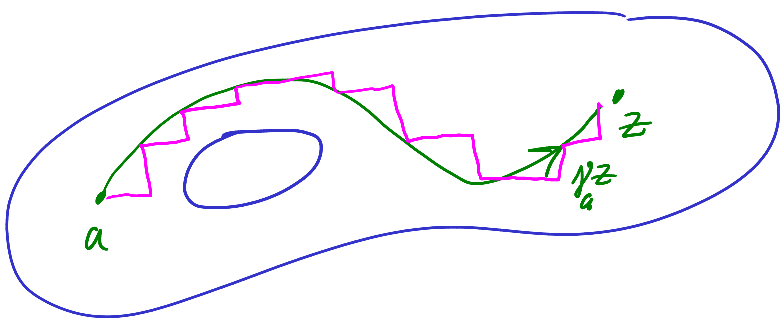
See Lecture 6 for a worked example.

4. Use the Cauchy-Riemann equations to prove that a real valued analtyic function on a domain must be constant.

$$f(x+iy) = u(x,y) + i\underbrace{v(x,y)}_{\equiv 0 \text{ if real valued}}$$

C-R eqns :
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \equiv 0 \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \equiv 0 \end{cases}$$

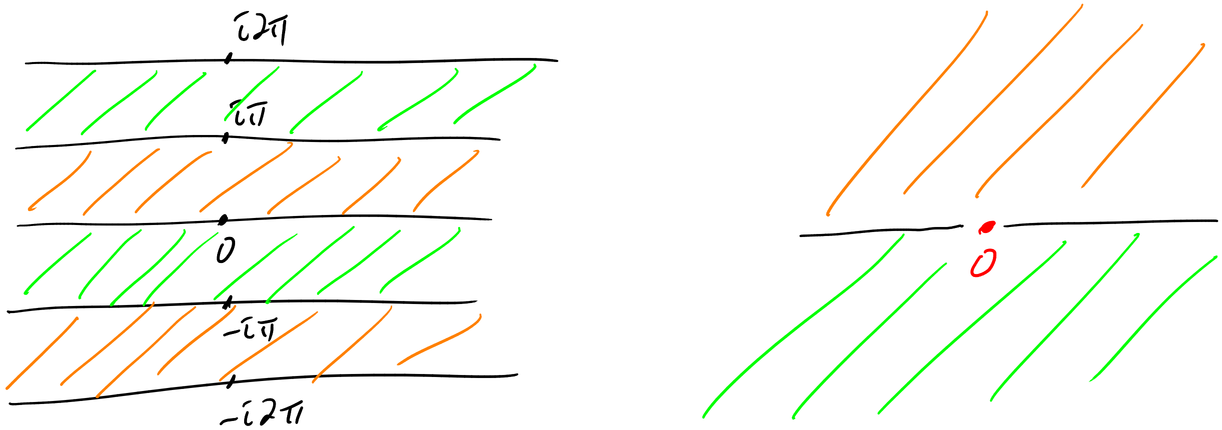
So $\nabla u \equiv 0$ on a domain

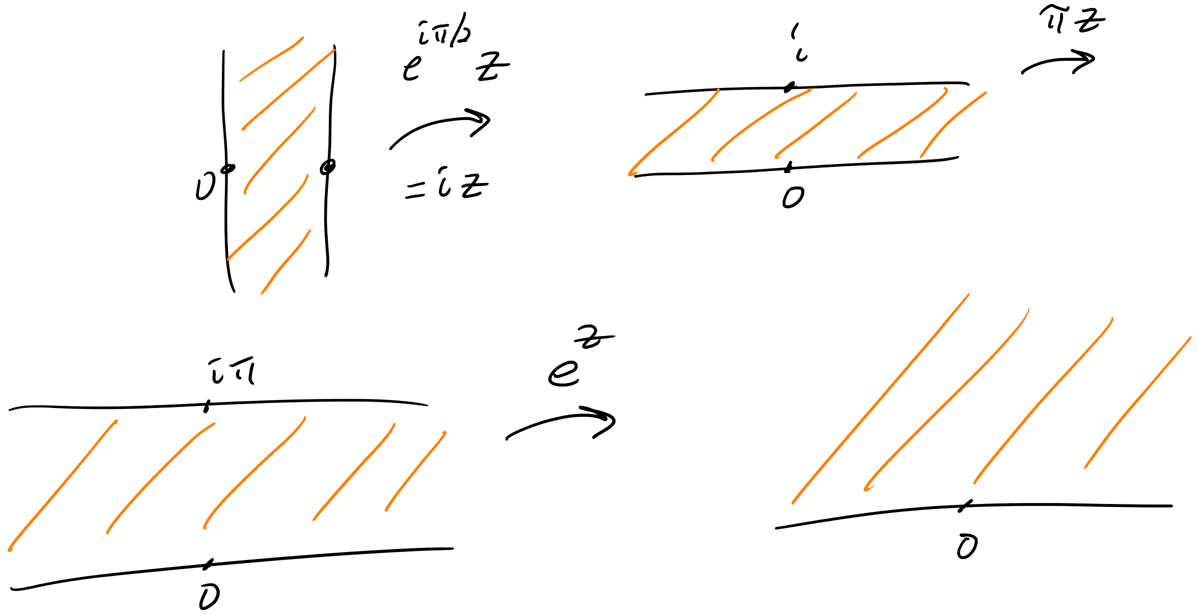


$$0 = \int_{\gamma_a^z} \underbrace{\nabla u}_{0} \cdot d\vec{s} = u(z) - u(a)$$

14. Find a one-to-one analytic function that maps the strip $\{z : 0 < \operatorname{Re} z < 1\}$ onto the upper half plane.

Think e^z

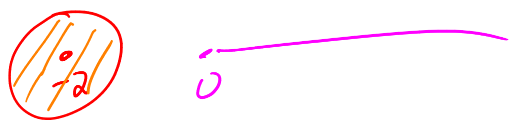




$f(z) = e^{i\pi z}$ does it.

13. Determine a branch of $\log(\underbrace{z^2 + 4z + 1})$ that is analytic near $z = -1$ and find its derivative there.

$(-1)^2 + 4(-1) + 1 = -2 \leftarrow$ Princ branch Log out!



$\log_0 z = \mathrm{Ln}|z| + i\theta$

$\theta \in \arg z,\ 0 < \theta < 2\pi$

$\log_0(z^2+4z+1)$ is analytic near $z=-1$.

$$\frac{d}{dz} = \frac{1}{(z^2+4z+1)} \cdot \left[2z+4\right] \qquad \text{chain rule.}$$

$$\frac{d}{dw} \log_\alpha w = \frac{1}{w} \qquad . \quad w = z^2+4z+1.$$