Lesson 2 The complex exponential

Def^n $e^z = e^{x+iy} = (e^x \cos y) + i(e^x \sin y)$

Properties

1) $e^0 = 1$
2) $\frac{d}{dz}(e^z) = e^z$

determine e^z

3) $e^{z+w} = e^z e^w$

"Group homomorphism: Additive grp $\mathbb{C}$ $\rightarrow$ mult grp $\mathbb{C}-\{0\}$"

4) $e^{-z} = 1/e^z$ $\leftarrow$ e^z is <u>never</u> zero

(3): $e^{z+(-z)} = e^z e^{-z} = e^0 = 1$

not zero.

5) DeMoivre's formula: $(e^{i\theta})^N = e^{iN\theta}$, $N \in \mathbb{Z}$

(3): $(e^{i\theta})^N = \underbrace{e^{i\theta} e^{i\theta} \cdots e^{i\theta}}_{N\text{-times}} = e^{i\theta + \cdots + i\theta} = e^{iN\theta}$

Pf of (3) **Step 1** $e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$

$$\cos(\alpha+\beta) + i \sin(\alpha+\beta) \stackrel{?}{=} (\cos\alpha + i \sin\alpha)(\cos\beta + i \sin\beta)$$

$$\stackrel{?}{=} \underbrace{(\cos\alpha \cos\beta - \sin\alpha \sin\beta)}_{\cos(\alpha+\beta)\ \checkmark} + i \underbrace{(\cos\alpha \sin\beta + \sin\alpha \cos\beta)}_{\sin(\alpha+\beta)\ \checkmark}$$

Step 2 $\quad e^{x+iy}\, e^{u+iv} = (e^{x} e^{iy})(e^{u} e^{iv})$

$$= (e^{x} e^{u})(e^{iy} e^{iv}) \leftarrow \text{associative law } \checkmark$$

$$= e^{x+u}\, e^{i(y+v)} = e^{(x+u)+i(y+v)} \quad \checkmark$$

Cool thing: Get trig identities via algebra!

Ex $\quad \sin 2\theta = 2 \sin\theta \cos\theta$

$$\sin 2\theta = \operatorname{Im}(e^{i2\theta})$$

$$e^{i2\theta} = (e^{i\theta})^2 = [\cos\theta + i \sin\theta]^2$$

$$= (\ \cdots\ ) + i\,\underbrace{(\cos\theta \sin\theta + \sin\theta\cos\theta)}_{= \sin 2\theta \ \checkmark}$$

Arithmetic on $\mathbb{C}$: Algebra – just like $\mathbb{R}$.

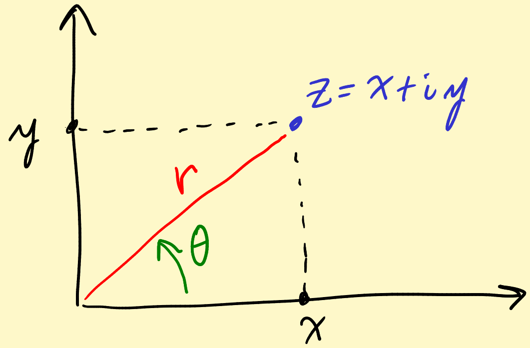
$\mathbb{C}$ is a field

$\mathbb{C}$ is "complete", meaning every Cauchy seq in $\mathbb{C}$ converges to a limit

$\mathbb{C}$ is "algebraically closed", meaning every poly in $z \in \mathbb{C}$ has a root in $\mathbb{C}$ (unlike $\mathbb{R}$!).

Analysis on $\mathbb{C}$: Topology same as $\mathbb{R}^2 \approx \mathbb{C}$

Complex #s in polar via e^{z}



$$x = \text{Re}\, z = r\cos\theta$$

$$y = \text{Im}\, z = r\sin\theta$$

$$r = |z| = \sqrt{x^2+y^2}$$

$$z = x+iy = r\cos\theta + i(r\sin\theta) = \boxed{re^{i\theta}} \leftarrow \text{polar form of } z$$

$\theta = \text{Arg}\, z \leftarrow$ Principal argument of z

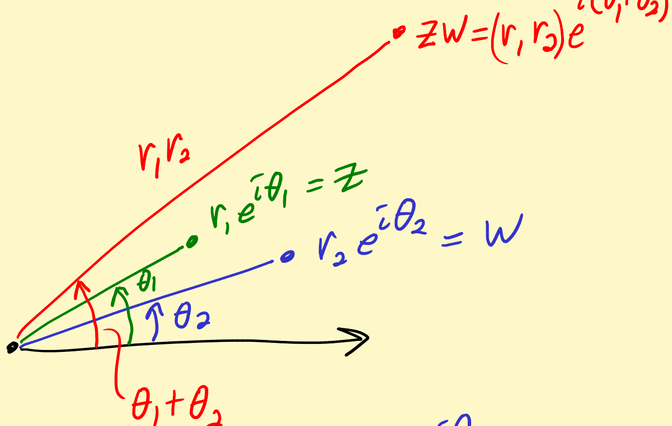
$$-\pi < \theta \le \pi$$

Def^n $\arg z$ = set of all θ with $z = re^{i\theta} \leftarrow r = |z|$

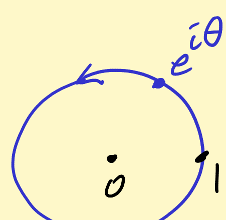
$$= \{\text{Arg}\, z + 2n\pi : n \in \mathbb{Z}\}$$

Adding complex #'s : vector addition in $\mathbb{R}^2$

Complex multiplication



Easy facts $|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$



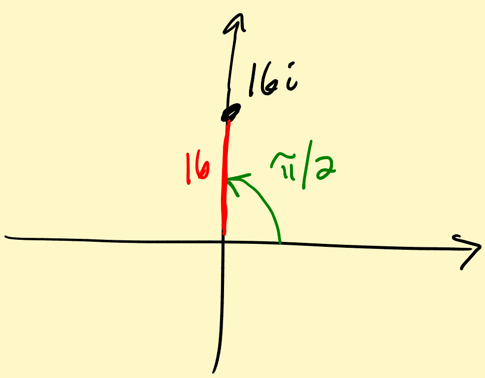
$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$
$$= \cos\theta - i\sin\theta = \overline{\left(e^{i\theta}\right)}$$

$$e^{i\theta} e^{-i\theta} = 1$$

$$\frac{1}{e^{i\theta}} = e^{-i\theta}$$ see DeMoivre's for $N<0$, $N \in \mathbb{Z}$.

Finding complex roots

Ex Find all $z \in \mathbb{C}$ such that $z^4 = 16i$



$$(\underbrace{re^{i\theta}}_{z})^4 = 16i$$ want

$$r^4 e^{i4\theta} = 16i$$

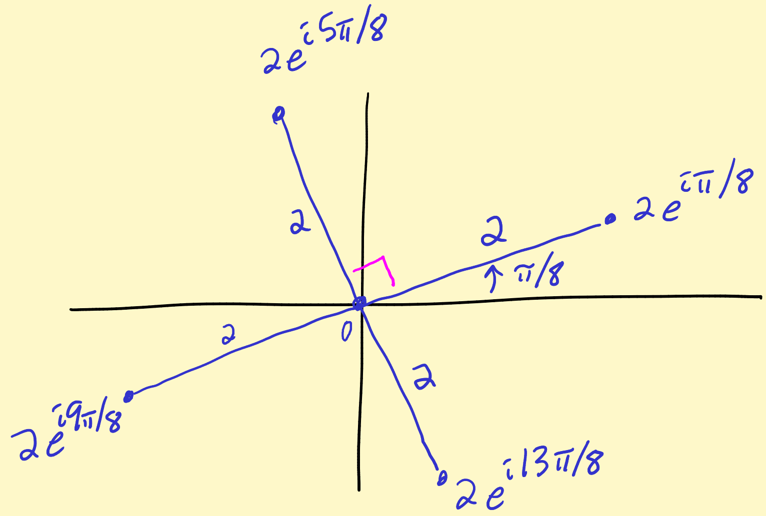
$$r^4 e^{i4\theta} = 16 e^{i\pi/2}$$ same # in $\mathbb{C}$

$\boxed{r^4 = 16}$ Note $r>0$ in polar form. So $r = 16^{1/4} = 2$.

Two polar forms need to represent the same complex #.

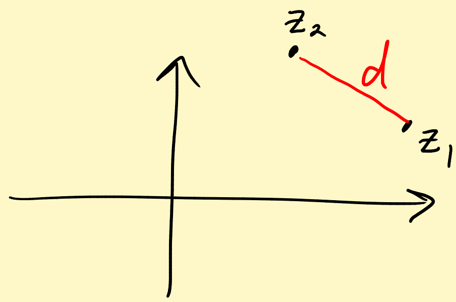
$$\boxed{4\theta = \frac{\pi}{2} + 2n\pi} \qquad n \in \mathbb{Z}$$

$$\theta = \frac{\pi}{8} + n\cdot\frac{\pi}{2} \qquad n\in\mathbb{Z}$$



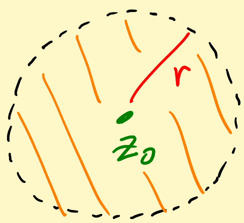
only $\underline{4}$ fourth roots of $16i$

Topology on $\mathbb{C} \approx \mathbb{R}^2$



$$d = \text{dist}(z_1, z_2) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} = |z_1 - z_2|$$

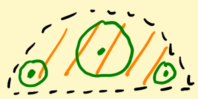
$$D_r(z_0) = \{ z : |z-z_0| < r \} \longleftarrow$$ open disc of radius r about z_0



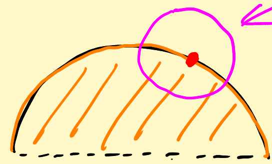
Def^n $\Omega \subset \mathbb{C}$ is called open if, for any $z_0 \in \Omega$, there is a disc $D_r(z_0)$ (with $r>0$) with $D_r(z_0) \subset \Omega$.

Note: r can depend z_0.

EX



open



ouch! Can't get disc in Ω about •

not open

Important Ω's: Connected open sets

path connected