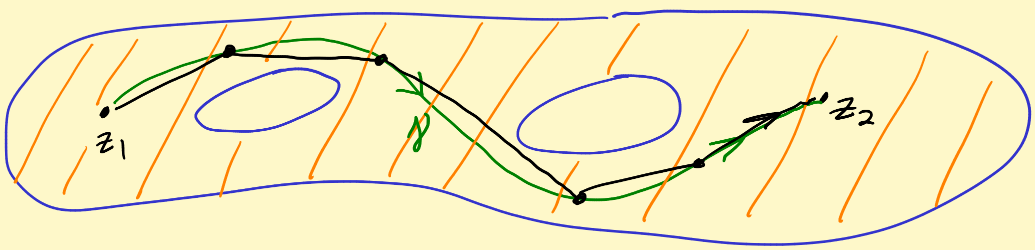
Lesson 3 Topology of $\mathbb{C}$, calculus on $\mathbb{C}$

No class Monday MLK Day

Def

Def^n An open set in $\mathbb{C}$ is (path) connected if any two pts in the set can be joined by a continuous curve in the set.



In fact, we can assume γ connecting z_1 to z_2 in Ω is piecewise C^1-smooth.

Def^n: Ω is a domain if it is open and (path) connected.

Lemma Suppose $u(x,y)$ is continuously diff'ble on a domain Ω and $\nabla u \equiv 0$ on Ω. Then $u \equiv$ const on Ω.

Pf: $\displaystyle\int_\gamma \underbrace{\nabla u}_{=0} \cdot d\vec{s} = u(z_2) - u(z_1) = 0$

↑ piecewise C^1

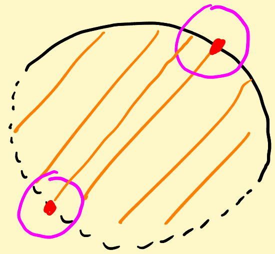
Hold z_1 fixed. Move z_2 around. ✓

Facts If Ω_1 and Ω_2 are open, then $\Omega_1 \cup \Omega_2$ is open and so is $\Omega_1 \cap \Omega_2$.

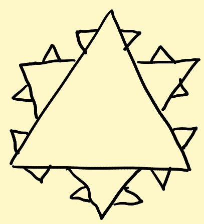
[Ω_j open. $\bigcup_{j=1}^{\infty} \Omega_j$ open, but $\bigcap_{j=1}^{\infty} \Omega_j$ might not be.]

Def^n A pt z_0 is a boundary pt of a set $S \subset \mathbb{C}$ if, for $\varepsilon > 0$ (no matter how small) $D_\varepsilon(z_0)$ contains a pt in S' and a pt outside S'.

EX



EX Serpinski snowflake is a domain in $\mathbb{C}$



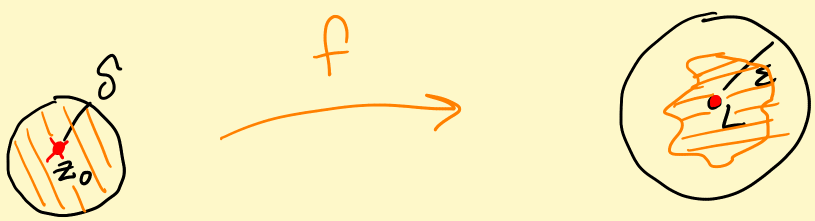
Finite area, but length of boundary curve is infinite!

Calculus on $\mathbb{C}$ Limits: $\lim_{z \to z_0} f(z) = L \in \mathbb{C}$ means:

Given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - L| < \varepsilon$$

when $|z - z_0| < \delta$ and $z \neq z_0$.



Def^n f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

(f cont on an open set means it is cont at every pt in the set.)

Fact All rules about limits from freshman calculus $\mathbb{R} \to \mathbb{R}$ carry over $\mathbb{C} \to \mathbb{C}$ with the same proofs!

EX $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then

$$\lim_{z \to z_0} f(z)g(z) = A \cdot B \quad \text{and} \quad \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$$

provided that $B \neq 0$.

Consequence Complex polys are continuous:

$$\text{Why:} \quad \underset{\text{cont}}{\underset{\uparrow}{a_0}} + \underset{\text{cont}}{\underset{\uparrow}{a_1}}\, \underset{\substack{\text{cont} \\ \varepsilon = \delta}}{\underset{\uparrow}{z}} + a_2 \underset{z \cdot z}{\underset{\uparrow}{z^2}} + a_3 \underset{z \cdot z^2}{\underset{\uparrow}{z^3}} \quad \text{etc.}$$

Def^n f is $\mathbb{C}$-diff'ble at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If it does, call limit $f'(z_0)$ $\left[\text{or } \left.\frac{df}{dz}\right|_{z = z_0}\right]$.

Fact All rules of diff $\mathbb{R} \to \mathbb{R}$ hold $\mathbb{C} \to \mathbb{C}$.

EX $(fg)' = f'g + fg'$

$$\frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} = \frac{f(z)g(z) - f(z_0)g(z) + f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0}$$

$$= \underbrace{g(z)}_{\downarrow} \cdot \underbrace{\frac{f(z)-f(z_0)}{z-z_0}}_{\downarrow} + \underbrace{f(z_0)}_{\downarrow} \cdot \underbrace{\frac{g(z)-g(z_0)}{z-z_0}}_{\downarrow}$$

$z \to z_0$: $\quad g(z_0) \cdot f'(z_0) + f(z_0) \cdot g'(z_0)$ ✓

Fact g $\mathbb{C}$-diff at $z_0 \Rightarrow g$ continuous at z_0.

Pf:
$$DQ = \frac{g(z)-g(z_0)}{z-z_0} = g'(z_0) + E(z)$$

where $E(z) \to 0$ as $z \to z_0$.

Solve for $g(z)$.
$$g(z) = g(z_0) + g'(z_0)(z-z_0) + E(z)\cdot(z-z_0)$$

$z \to z_0$:
$$g(z_0) + g'(z_0)\cdot 0 + 0 \cdot 0$$

$$g(z) \to g(z_0) \text{ as } z \to z_0. \checkmark$$

Def^n f is analytic on an open set $\Omega \subset \mathbb{C}$ if it is $\mathbb{C}$-diff'ble at each pt in Ω.

MA 425: Analytic fcns on domains in $\mathbb{C}$.

Fact Complex polys are analytic on $\mathbb{C}$.

EX: 1) $z^2 = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$

2) $e^z = e^{x+iy} = \underbrace{(e^x \cos y)}_{u(x,y)} + i\,\underbrace{(e^x \sin y)}_{v(x,y)}$

Is e^z analytic. Yes! But need more than freshman calc to see.

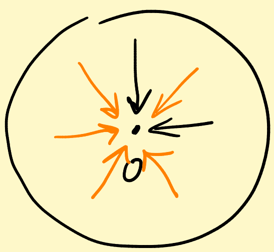
3) $\bar{z} = x - iy$ is <u>not</u> $\mathbb{C}$-diff'ble!

$$DQ = \frac{\bar{z} - \bar{0}}{z - 0} = \frac{\bar{z}}{z}$$

$z = re^{i\theta}$

$z \to 0$ exactly when $r \to 0$

$$= \frac{\overline{(re^{i\theta})}}{re^{i\theta}} = \frac{r e^{-i\theta}}{r e^{i\theta}} = e^{-i2\theta}$$



all different!

Soon: See Cauchy-Riemann equations.