# Lecture 4 Complex power tools!

HWK 1 due in Gradescope tomorrow
Thurs 11:30 pm
No late papers without prior permission

**Geometric series**

$$\underbrace{1+z+z^2+\cdots+z^N}_{S_N(z)} = \frac{1}{1-z} - \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

$$- \quad z\,S_N(z) = z+z^2+\cdots+z^{N+1}$$

$$1 - z^{N+1}$$

$$(1-z)\,S_N(z) = 1 - z^{N+1} \quad \checkmark$$

**Fact** If $|z|<1$, then $S_N(z) \to \frac{1}{1-z}$ as $N\to\infty$.

Numerator estimate: $|z+w| \le |z| + |w|$

Denominator estimate: $|z-w| \ge \big||z|-|w|\big|$

also $|z+w| \ge \big||z|-|w|\big|$ replace $w$ with $-w$

**EX** $$\left|\frac{a+b}{c+d}\right| \le \frac{|a|+|b|}{\big||c|-|d|\big|}$$

**Pf of fact** $$\left|S_N(z) - \frac{1}{1-z}\right| = \left|\frac{-z^{N+1}}{1-z}\right|$$

$$\le \frac{|z^{N+1}|}{\big|1-|z|\big|}$$

If $|z|<1$, then $\big|1-|z|\big| = 1-|z|$

$$\le \frac{|z|^{N+1}}{1-|z|} \quad \text{when } |z|<1$$

$\to 0$ as $N \to \infty$ when $|z|<1$.

Calculus fact $\mathbb{C} \to \mathbb{C}$ same as $\mathbb{R} \to \mathbb{R}$:

If a series converges, the terms have to $\to 0$. (same pf)

Consequently, if $|z| \geq 1$, geometric series does not converge

since $|z^N| = |z|^N \geq 1^N = 1$ shows terms do not $\to 0$ as $N \to \infty$.

Power of the complex exponential

D'M: $(e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta \qquad n \in \mathbb{Z}$

Famous formulas:

$$
\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta \\
+ \quad e^{-i\theta} &= \cos\theta - i\sin\theta \\
\hline
e^{i\theta} + e^{-i\theta} &= 2\cos\theta
\end{aligned}
$$

$$\boxed{\begin{aligned} \cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i} \end{aligned}}$$

$(-)$

Soon, let $\theta \in \mathbb{C}$. Use formula to define $\cos, \sin$ for "angles" in $\mathbb{C}$.

Fourier series fact

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos N\theta = \frac{\sin[(N+\frac{1}{2})\theta]}{2\sin\frac{\theta}{2}}$$

Dirichlet Kernel $D_N(\theta)$.

EX $\quad 1 + \cos\theta + \cdots + \cos N\theta =$

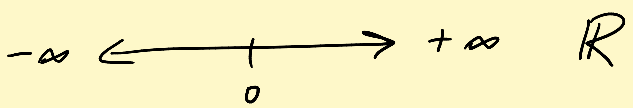
$$\mathrm{Re}\Big[\underbrace{1 + e^{i\theta} + \cdots + e^{iN\theta}}_{1+(e^{i\theta})+(e^{i\theta})^2+\cdots+(e^{i\theta})^N}\Big] =$$

$$= \frac{1-(e^{i\theta})^{N+1}}{1-e^{i\theta}} \cdot \frac{(-e^{i\theta/2})/2i}{(-e^{-i\theta/2})/2i}$$

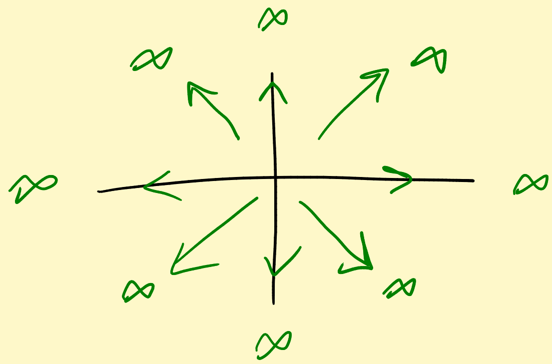
$$= \frac{\ell\ell\ell}{\dfrac{e^{i\theta/2}-e^{-i\theta/2}}{2i}} \longleftarrow = \sin\frac{\theta}{2}$$

$$\Sigma = \frac{\mathrm{Re}[\ell\ell\ell]}{\sin\frac{\theta}{2}}$$ clean, simple formula

The "point at infinity"




$\mathbb{C}$




Def$^n$ $\lim_{z\to a} f(z) = \infty$

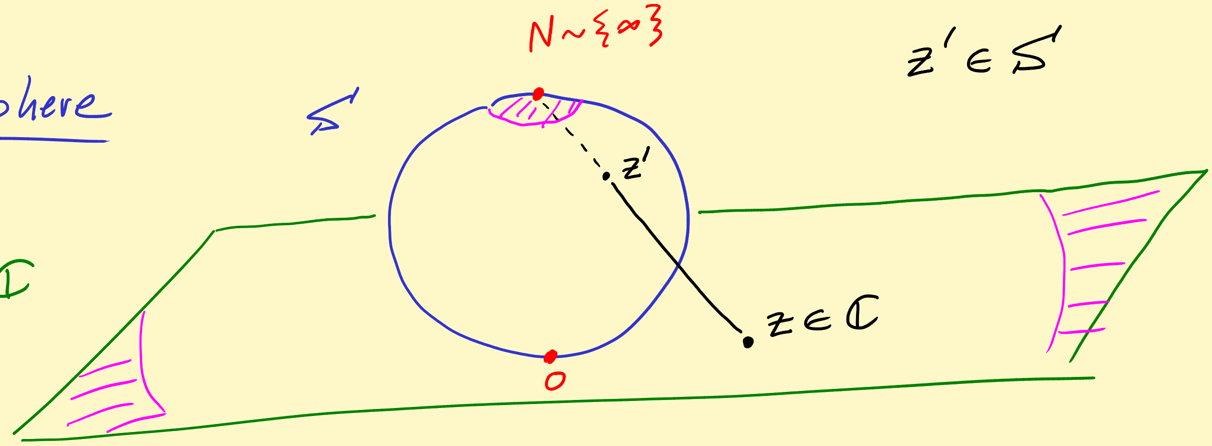
means $|f(z)| \to +\infty$

as $z \to a$.

More precisely, given $M > 0$, there is a $\delta > 0$ such that $|f(z)| > M$ when $|z-a| < \delta$, $z \neq a$.

EX: $\lim_{z \to a} \frac{1}{z-a} = \infty \qquad \lim_{z \to a} \frac{1}{(z-a)^3} = \infty.$

## Riemann sphere




$[-\infty, \infty]$ "compactifies" $\mathbb{R}$.

$\underbrace{\mathbb{C} \cup \{\infty\}}$ is the "one-point compactification" of $\mathbb{C}$

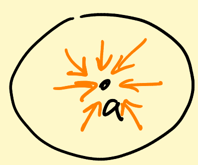
$\hat{\mathbb{C}}$ extented complex plane.

$D_\varepsilon(\infty)$ = "polar cap" = $\{z \in \mathbb{C} : |z| > \frac{1}{\varepsilon}\} \cup \{\infty\}$

small neighborhood (or disc) about $\infty$.

Coming next: Cauchy-Riemann equns

EX: $z^2$: $\quad DQ = \frac{z^2 - a^2}{z-a} = z + a \longrightarrow 2a$ as $z \to a$.




EX: $\bar{z}$: $\quad DQ = \frac{\bar{z} - \bar{a}}{z-a} \qquad z = a + re^{i\theta}$

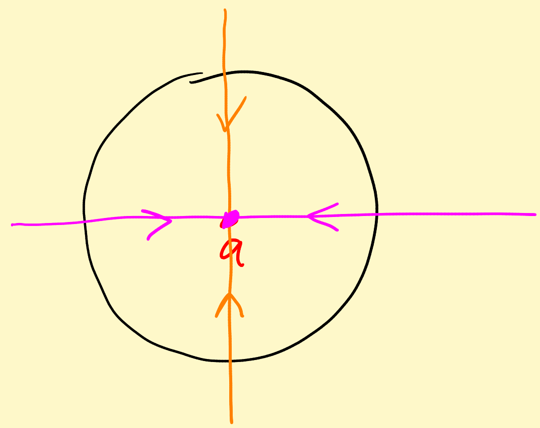
$$= \frac{\overline{(re^{i\theta})}}{re^{i\theta}} = e^{-2i\theta}$$

$z \to a$ when $r \to 0$. Different limits along radial directions.

$\lim_{z \to a}$ DQ does not exist. So $\bar{z}$ is not complex diff'ble at any pt $a \in \mathbb{C}$.

Cauchy-Riemann equs $\quad f(x+iy) = u(x,y) + i v(x,y)$

$$DQ = \frac{f(z) - f(a)}{z - a}$$




If $f'(a)$ exists,

horiz and vert limits exist and are same.

Exciting fact. Reverse is true!

Cauchy-Riemann equs (C-R equs)

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$\bar{z} = \overline{x+iy} = x - iy$

$u(x,y) = x \quad v(x,y) = -y$

C-R equs fail!