Lecture 5 The Cauchy-Riemann equations (C-R equs) Read Chap 2

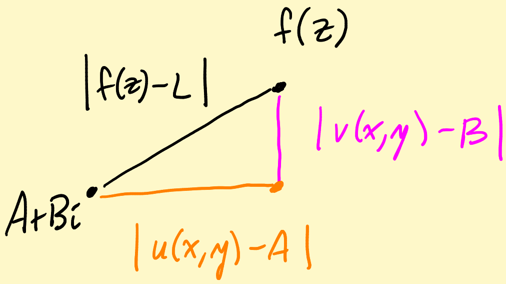
Lemma Write $f(x+iy) = u(x,y) + i v(x,y)$

$$z = x+iy \qquad z_0 = x_0 + i y_0$$

$$\lim_{z \to z_0} f(z) = \underbrace{A + iB}_{L} \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = A$$

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = B$$

Why



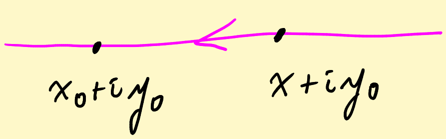
See lemma from

$\Rightarrow$ legs $\leq$ hyp

$\Leftarrow$ Pythagorean

Cauchy-Riemann equs : $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

Step 1 Suppose $f(z)$ is $\mathbb{C}$-diff'ble at z_0.



$$DQ = \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \frac{\left[u(x,y_0) + i v(x,y_0)\right] - \left[u(x_0,y_0) + i v(x_0,y_0)\right]}{(x+iy_0) - (x_0+iy_0)}$$

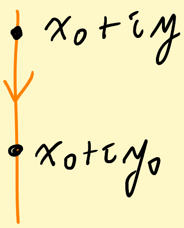
$$= \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i \, \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}$$

Aha! If $DQ \to f'(z_0)$, then lemma shows that real and imag parts $\to$ limits. See that first partials in x <u>exist</u> and

$$\boxed{f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0)}$$

Red box #1

Next



$$DQ = \frac{f(x_0+iy) - f(x_0+iy_0)}{(x_0+iy) - (x_0+iy_0)}$$

$$= \frac{u(x_0,y) - u(x_0,y_0)}{i\,(y-y_0)} + \frac{v(x_0,y) - v(x_0,y_0)}{y-y_0}$$

$\frac{1}{i} = -i$

$$\longrightarrow \frac{\partial v}{\partial y}(x_0,y_0) \;-\; i\,\frac{\partial u}{\partial y}(x_0,y_0)$$

Conclude first partials in y <u>exist</u> and

$$\boxed{f'(z_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i\,\frac{\partial u}{\partial y}(x_0,y_0)}$$

Red box #2

Equate red boxes to get C-R equs.

<u>Thm</u> If $f = u+iv$ is $\mathbb{C}$-diff'ble at z_0, then the first partials of u and v exist at (x_0,y_0) and

$$f'(z_0) = \begin{cases} u_x + i v_x \\ \\ v_y - i u_y \end{cases} \quad \text{at } (x_0, y_0) \quad \left(\overset{\text{see}}{\text{C-R equs}}\right)$$

EX $f(z) = \bar{z} = x - iy$ $\quad u(x,y) = x \quad v(x,y) = -y$

C-R equs $\quad \frac{\partial u}{\partial x} = 1 \overset{?}{=} -1 = \frac{\partial v}{\partial y}$ No

$\frac{\partial u}{\partial y} = 0 \overset{?}{=} 0 = -\frac{\partial v}{\partial x}$ Yes, but who cares!

f' exists $\Rightarrow$ both C-R equs hold. They don't.

So $\bar{z}$ is not $\mathbb{C}$-diff'ble.

EX: $\quad e^{x+iy} = \underbrace{(e^x \cos y)}_{u} + i \underbrace{(e^x \sin y)}_{v}$

The C-R equs hold! Is e^z is $\mathbb{C}$-diff'ble?

We showed: f' exists $\Rightarrow$ C-R equs hold.

Is reverse true? Yes! with some minimal further assumptions about u, v. [Need u,v to be C^1-smooth]

Thm If u and v are C^1-smooth on an open set Ω

[meaning u, v are continuous on Ω, all first partials exist and are also continuous on Ω], and u, v satisfy the C-R equns, $f = u + iv$ is <u>analytic</u> on Ω (f is $\mathbb{C}$-diff'ble on <u>open</u> set Ω)

<u>Calculus lemma</u> If $u(x,y)$ is C^1-smooth near (x_0,y_0), then

$$u(x,y) = \underbrace{u(x_0,y_0) + u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0)}_{\text{first order Taylor poly}} + R(x,y)$$

↑ remainder

where $$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Assume lemma. <u>Pf of Thm</u> Notation $u^0 = u(x_0,y_0)$

$u_x^0 = u_x(x_0,y_0)$ etc

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{(u+iv)-(u^0+iv^0)}{z-z_0}$$

C-R equns

$$= \frac{\left[u_x^0(x-x_0) + \underbrace{u_y^0}_{=-v_x^0}(y-y_0)\right] + i\left[v_x^0(x-x_0) + \underbrace{v_y^0}_{=u_x^0}(y-y_0)\right]}{z-z_0} + \frac{R_u + iR_v}{z-z_0}$$

Miracle! See complex multiplication!

Red box #1

$$= \frac{\boxed{u_x^0 + i v_x^0} \cdot \overbrace{\left[(x-x_0) + i(y-y_0)\right]}^{z-z_0}}{z-z_0} + \mathcal{R}$$

$$= \left(u_x^0 + i v_x^0\right) + \mathcal{R}$$

where $|\mathcal{R}| = \left|\frac{R_u + i R_v}{z-z_0}\right| \leq \frac{|R_u|}{\sqrt{\quad}} + \frac{|R_v|}{\sqrt{\quad}}$

$\longrightarrow 0$ as $(x,y) \to (x_0,y_0)$. Done!

<u>Cor</u> e^z is analytic on $\mathbb{C}$ and $\frac{d}{dz} e^z = e^z$.

<u>Check</u> $\frac{d}{dz} e^z =$ Red box #1 $= \frac{\partial}{\partial x}\left(e^x \cos y\right) + i \frac{\partial}{\partial x}\left(e^x \sin y\right)$

$$= e^x \cos y + i e^x \sin y \quad \checkmark$$

<u>Remark</u> $\mathbb{R}^2 \to \mathbb{R}^2$

$$(x,y) \mapsto \left(u(x,y), v(x,y)\right)$$

Jacobian matrix: $J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$

$$= \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \leftarrow \text{C-R eqns}$$

$J : \mathbb{R}^2 \xrightarrow{\text{linear}} \mathbb{R}^2$

C-R eqns $\Rightarrow$ linear trans represents complex multiplication
by $A + Bi$

Think about doing C-R eqns twice.

Amazing fact! If f is analytic, then so is f'!

(way false $\mathbb{R} \to \mathbb{R}$.)

f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analtytic $\Rightarrow \cdots$

able to conclude that u, v are C^∞-smooth

and that u and v are harmonic functions, meaning

that $\Delta u = 0$, $\Delta v = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$