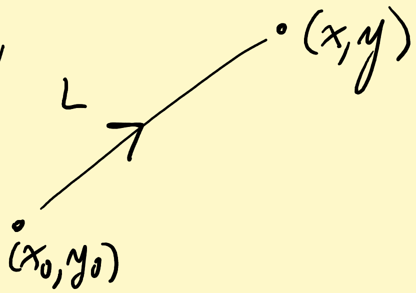
Lecture 6 Cauchy-Riemann equations, applications

Calculus lemma Suppose $u(x,y)$ C^1-smooth. Then

$$u(x,y) = \underbrace{\overbrace{u(x_0,y_0)}^{u^0} + \overbrace{\frac{\partial u}{\partial x}(x_0,y_0)}^{u_x^0}(x-x_0) + \overbrace{\frac{\partial u}{\partial y}(x_0,y_0)}^{u_y^0}(y-y_0)}_{\text{First order (linear) Taylor approx to } u} + \underbrace{R(x,y)}_{\text{remainder}}$$

where
$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Why



$$L: (x_0,y_0) + t(x-x_0, y-y_0)$$
$$0 \le t \le 1$$

$$u(x,y) - u(x_0,y_0) = \int_L \nabla u \cdot d\vec{s} =$$

$$= \int_0^1 \left(\frac{\partial u}{\partial x}\big(\underbrace{x_0+t(x-x_0), y_0+t(y-y_0)}\big), \frac{\partial u}{\partial y}\big(x_0+t(x-x_0), y_0+t(y-y_0)\big) \right)$$

$$\cdot\ (x-x_0, y-y_0)\, dt$$

$$= \int_0^1 \frac{\partial u}{\partial x}\cdot(x-x_0) + \frac{\partial u}{\partial y}\cdot(y-y_0)\, dt$$

Hmmm. $$\underbrace{\left(u_x^0\cdot(x-x_0)+u_y^0\cdot(y-y_0)\right)}_{\text{constant}}=\int_0^1\left(u_x^0\cdot(x-x_0)+u_y^0\cdot(y-y_0)\right)dt$$

$$R(x,y)=\int_0^1 (u_x-u_x^0)(x-x_0)+(u_y-u_y^0)(y-y_0)\,dt$$

$$|R(x,y)|\le\underbrace{\left(\operatorname*{Max}_L|u_x-u_x^0|\right)}_{\text{Make }<\frac{\varepsilon}{2}}\cdot|x-x_0|+\underbrace{\left(\operatorname*{Max}_L|u_y-u_y^0|\right)}_{\text{Make }<\frac{\varepsilon}{2}}\cdot|y-y_0|$$

Let $\varepsilon>0$.

$$\frac{|R(x,y)|}{\sqrt{}}\le\frac{\varepsilon}{2}\underbrace{\frac{|x-x_0|}{\sqrt{}}}_{\le 1}+\frac{\varepsilon}{2}\underbrace{\frac{|y-y_0|}{\sqrt{}}}_{\le 1}<\varepsilon\ \checkmark$$

[(leg) ≤ (hyp)]

<u>Exiting consequence of C-R eqns</u> Suppose u,v C^2-smooth real valued fcns on $\mathbb{R}^2$ that satisfy the C-R eqns (so $u+iv$ is analytic). Then u and v are harmonic!

<u>Why</u>
$$\begin{cases} u_x=v_y & (A)\\ u_y=-v_x & (B)\end{cases}$$

$\frac{\partial}{\partial x}(A)$: $u_{xx}=v_{xy}$

$\frac{\partial}{\partial y}(B)$: $u_{yy}=-v_{yx}$

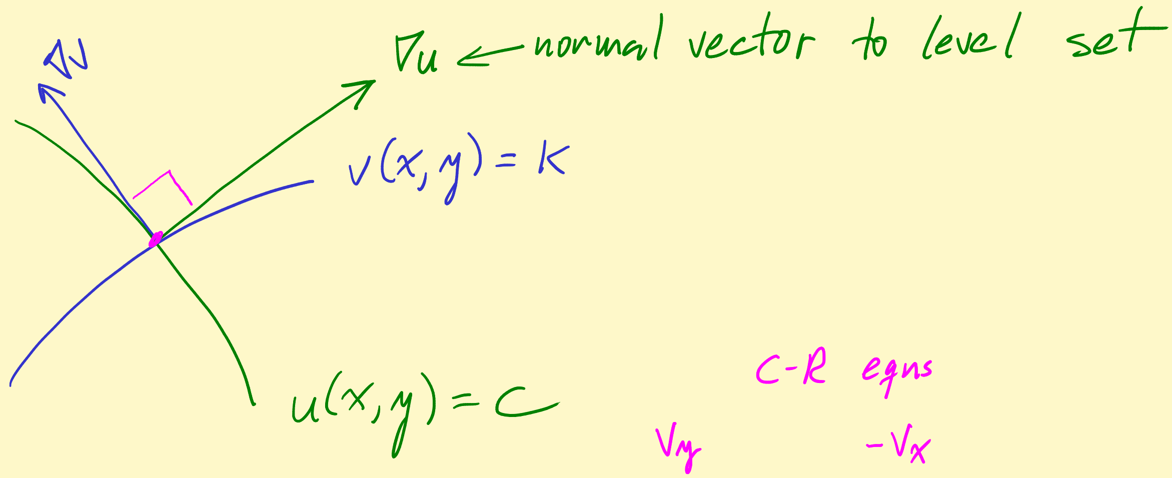
$+$: $u_{xx}+u_{yy}=0$ $\checkmark$

v C^2-smooth so $v_{xy}=v_{yx}$

Similarly $\frac{\partial}{\partial y}(A) - \frac{\partial}{\partial x}(B)$ shows v harmonic too.

Big facts f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic $\Rightarrow \cdots$ implies that u, v are C^∞-smooth, so f analytic $\Rightarrow$ u, v harmonic.

Great fact Level sets of u, v are $\perp$.



C-R eqns

$$\nabla u \cdot \nabla v = \begin{pmatrix} u_x \\ u_y \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \overset{v_y}{\overset{\shortparallel}{u_x}} v_x + \overset{-v_x}{\overset{\shortparallel}{u_y}} v_y = 0$$

Normal vectors $\perp$ $\Rightarrow$ Tangent vectors are too.

Question Given a C^2-smooth harmonic fcn u, can I find a v such that $u + iv$ analytic.

[yes, on open sets with no holes.]

Hmmm. Given u, want v with

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad \begin{matrix} v_x = -u_y \\ v_y = u_x \end{matrix} \quad \text{given}$$

Want v with $\nabla v = \vec{F}$ where $\vec{F} = \begin{pmatrix} -u_y \\ u_x \end{pmatrix}$.

Want a "potential fcn" v for field $\vec{F}$.

In $\mathbb{R}^3$, need Curl $\vec{F} = 0$ to get v.

In $\mathbb{R}^2$: think $\nabla \varphi = (\varphi_x, \varphi_y) = (F_1, F_2)$

Know $\varphi_{yx} = \varphi_{xy}$.

$$\frac{\partial}{\partial y}(\underbrace{\varphi_x}_{F_1}) = \frac{\partial}{\partial x}(\underbrace{\varphi_y}_{F_2})$$

$$(*) \quad \boxed{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}}$$

necessary cond to get φ.

MA 261: Condition is also sufficient in domains without holes.

Do we have $(*)$? $\qquad \vec{F} = \begin{pmatrix} -u_y \\ u_x \end{pmatrix} \begin{matrix} \leftarrow F_1 \\ \leftarrow F_2 \end{matrix}$

$$\frac{\partial}{\partial y}(-u_y) \stackrel{?}{=} \frac{\partial}{\partial x}(u_x)$$

yes! $\quad u_{xx} + u_{yy} \equiv 0 \quad \checkmark$

Can find v!

EX $\quad u(x,y) = x^2 - y^2 + xy + 3 \quad$ Check: $\Delta u \equiv 0$ ✓

Find v with
$$\begin{cases} v_x = -u_y = -\frac{\partial}{\partial y}(x^2 - y^2 + xy + 3) = 2y - x & \text{(A)} \\ v_y = u_x = 2x + y & \text{(B)} \end{cases}$$

$$\boxed{\begin{aligned} &(A): \quad v_x = 2y - x \\ &(B): \quad v_y = 2x + y \end{aligned}}$$

Step 1 Use (A)

$$V = \int 2y - x \, dx = (2y)x - \frac{x^2}{2} + \underbrace{C(y)}_{\substack{\text{arb} \\ \text{fcn of } y \\ \text{(the other var)}}}$$

treat y's like constants
"partial integral"

Got $\boxed{v = 2xy - \frac{1}{2}x^2 + C(y)}$ Done with (A)

Step 2 Use (B), but in a different way.

(B): $\quad V_y = 2x + y$

$$\frac{\partial}{\partial y}\boxed{2xy - \tfrac{1}{2}x^2 + C(y)} \overset{\text{want}}{=} 2x + y$$

$$2x - 0 + C'(y) = 2x + y$$

Miracle! All x's cancel! [If they don't, you made an error (or maybe u was <u>not</u> harmonic).]

$$\boxed{C'(y) = y}$$

so $$C(y) = \int y\,dy = \frac{1}{2}y^2 + K$$

Go back to box $\boxed{v = \cdots + C(y)}$ and fill in $C(y)$.

Get $$v = 2xy - \frac{1}{2}x^2 + \underbrace{\frac{1}{2}y^2 + K}_{C(y)}$$

<u>Defn</u> v is a <u>harmonic conjugate</u> for u.

<u>Fact</u> v is <u>unique</u> up to addition of a constant.

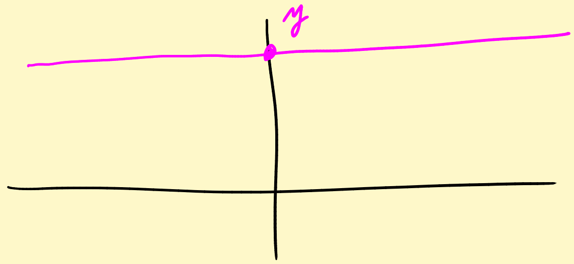
Hmmm. $u+iv$ analytic. How to write as a fcn of z?

$$\begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases} \qquad x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$$

plug into $u+iv$ above. Miracle: all $\bar{z}$'s cancel out leaving fcn of z alone.

More about why the C is a $C(y)$.

Fact: If $\frac{\partial v}{\partial x} \equiv 0$, then $v =$ fcn of y alone.



$\leftarrow$ v is const along horiz lines. Call it $C(y)$. ✓

Conversely $\frac{\partial}{\partial x}[C(y)] \equiv 0$.

Our v above: $\int$ calculation produced v with the $\frac{\partial v}{\partial x}$ we wanted it to have. If $\tilde{v}$ was any other such v, then

$$\frac{\partial}{\partial x}\Big[\underbrace{v - \tilde{v}}_{\text{must be } = C(y)}\Big] = 0.$$

See $\tilde{v} = v + C(y)$ ✓