Lecture 7 Analytic functions, more on complex exponential

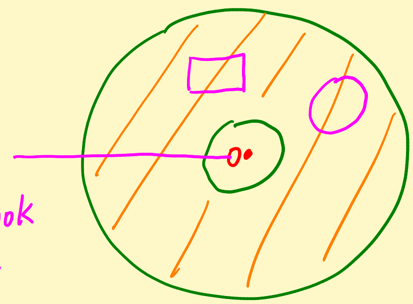
Warning If a domain has holes, harmonic conjugate might not exist (on whole domain).

Last time: showed how to cook up harmonic conj on rectangle or disc.

EX $u = \operatorname{Ln}\sqrt{x^2+y^2} = \operatorname{Ln}|z|$. On a disc or restangle $\left[(-\infty, 0] \text{ not in}\right]$

$$v = \operatorname{Arg}(x+iy) + K = \operatorname{Tan}^{-1}\frac{y}{x} + K$$

ouch! Can't hook up v's.



Basic facts about analytic fcns: complex diff'ble fcns on an open set.

1) Analytic fcns must be continuous. (done last week)

2) Rules of differentiation $\mathbb{R} \to \mathbb{R}$ hold $\mathbb{C} \to \mathbb{C}$

3) $\bar{z}$ is not complex diff'ble. (C-R eqns fail)

EX $|z|^2 + z^3 = f(z)$. Is f $\mathbb{C}$-diff'ble?

$$f(z) = z \cdot \bar{z} + z^3$$

Hmmm. If f were $\mathbb{C}$-diff, then so is $f(z) - z^3 = z \cdot \bar{z}$.

Aha! … so $\dfrac{f(z)-z^3}{z} = \bar{z}$ is too (when $z \neq 0$).

$\bar{z}$ is <u>not</u>. So f can't be when $z \neq 0$. [f is $\mathbb{C}$-diff at $z=0$. Write DQ.]

[Book 2.3: Song and dance about complex fcns that are "admissible": giving feeling of a fcn of z with no $\bar{z}$'s. Not standard.]

4) If f analytic on a domain Ω and $f'(z) \equiv 0$, then f is <u>constant</u> on Ω.

Better proof than last week in spirit of "complex analysis".

$f(x+iy) = u(x,y) + i\,v(x,y)$

Know: u, v continuous and first partials <u>exist</u> and satisfy C-R eqns

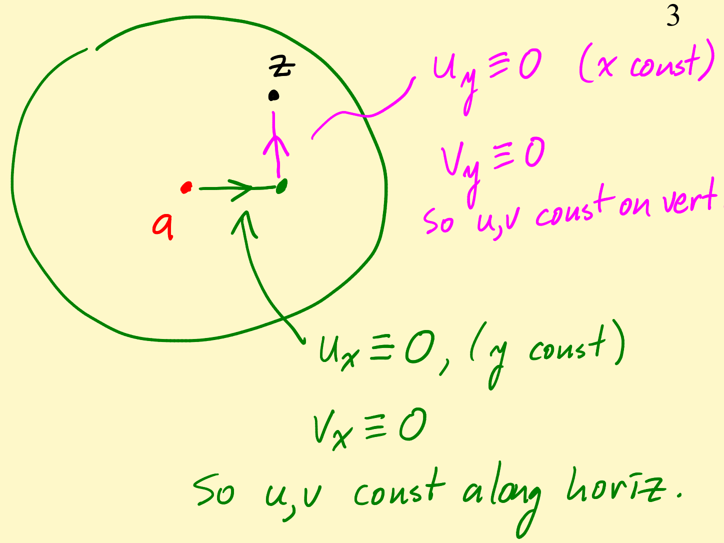
$$f'(z) = \begin{cases} u_x + i v_x & \#1 \\ v_y - i u_y & \#2 \end{cases}$$

$f'(z) \equiv 0 \implies \nabla u \equiv 0, \ \nabla v \equiv 0$

[Last time: assumed first partials were <u>continuous</u>. Don't have to.]

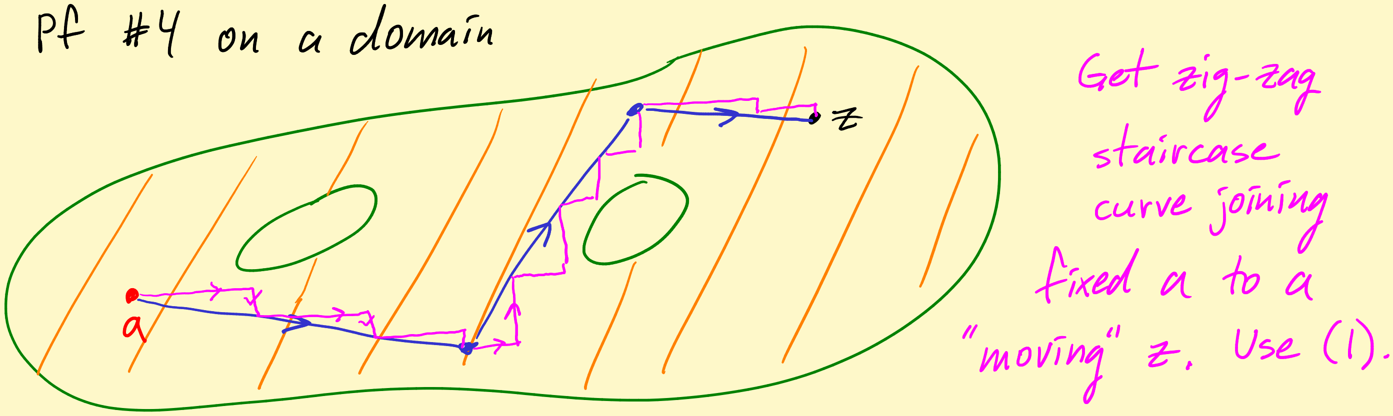
<u>Freshman calc fact</u> If $h(t)$ is continuous on $[a,b]$ and $h'(t)$ exists on (a,b) and $\equiv 0$ there, then h is <u>constant</u> on $[a,b]$. <u>Pf</u> MVT $\dfrac{h(t_2)-h(t_1)}{t_2-t_1} = h'(\tau)$

(2) pf of #4 on a disc:



Calc lemma => u,v at z same as at a. ✓

(2) Pf #4 on a domain



<u>Thm</u> Complex chain rule: $h(z) = f(g(z))$, $w_0 = g(z_0)$. If g is $\mathbb{C}$-diff'ble at z_0 and f is $\mathbb{C}$-diff'ble at w_0, then h is $\mathbb{C}$-diff'ble at z_0 &

$$h'(z_0) = f'(g(z_0))\, g'(z_0).$$

<u>Pf</u> $$\frac{f(w) - f(w_0)}{w - w_0} = f'(w_0) + E(w)$$ where $E(w) \to 0$ as $w \to w_0$

$$\boxed{f(w) - f(w_0) = f'(w_0)(w - w_0) + E(w)(w - w_0)}$$

Let $w = g(z)$. $[w_0 = g(z_0)]$

$$\frac{f(g(z)) - f(g(z_0))}{z - z_0} = f'(g(z_0))\,\frac{(g(z) - g(z_0))}{z - z_0} + E(g(z))\,\frac{(g(z) - g(z_0))}{z - z_0}$$

$$DQ \longrightarrow f'(g(z_0)) \cdot g'(z_0) + 0 \cdot g'(z_0)$$

Why $E(g(z)) \to 0$: g $\mathbb{C}$-diff at $z_0 \Rightarrow g$ cont at z_0.

Rule of limits : $E(w) \to 0$ as $w \to w_0$

$g(z) \to w_0 = g(z_0)$ as $z \to z_0$.

So $E(g(z)) \to 0$ as $w \to w_0$.

Careful: Should have <u>defined</u> $E(w_0) = 0$ to make it a <u>continuous</u> fcn so I could use rule of limits even in case $g(z) = w_0$ when $z \neq z_0$.

<u>Classic problem</u> Cook up complex exponential $E(z)$.

Function with 1) $E(0) = 1$

2) $E'(z) = E(z)$

$$E'(z) = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} \overset{\text{want}}{=} u + iv$$

(A): $u_x = u$ $\qquad$ $v_x = v$

$$\boxed{u = C(y)e^x} \qquad \boxed{v = K(y)e^x}$$

(B): $v_y = u$ $\qquad$ $-u_y = v$

$$K'(y)e^x = \underset{C(y)e^x}{u} \qquad -C'(y)e^x = \underset{K(y)e^x}{v}$$

$$\boxed{K'(y) = C(y)} \quad (1) \qquad \boxed{-C'(y) = K(y)} \quad (2)$$

(1)': $K''(y) = C'(y) \overset{\text{from (2)}}{=} -K(y)$

$$\boxed{K'' + K = 0} \qquad \underline{K(y) = A\cos y + B\sin y}$$

(1): $\underline{C(y) = K'(y) = -A\sin y + B\cos y}$.

Final thing. Use $E(0)=1$. Initial cond for ODE's pin down A, B.

Get $E(x+iy) = (e^x\cos y) + i\,(e^x\sin y)$!