Lecture 8 More about the complex exponential

Today: write $E(x+iy) = (e^x \cos y) + i(e^x \sin y)$

(After today, write e^{x+iy} for this.)

Properties

1) $E(0)=1$ ✓
2) $E'(z)=E(z)$ ✓
3) $E(z+w)=E(z)E(w)$
4) $E(-z)=1/E(z)$
5) $E(z)$ never zero
6) $E(z)=1 \iff z=n(2\pi i)$, $n\in\mathbb{Z}$

Lemma If f is analytic on $\mathbb{C}$ (or $D_R(0)$) and

$f'(z)=kf(z)$, then $f(z)=C\cdot E(kz)$ where $C=f(0)$.

Why

$$f' - kf = 0$$

First order linear ODE in $\mathbb{C}$ with const coef.

MA 262: Integrating factor e^{-kx}

Hmmm. Multiply by $E(-kz)$:

$$\underbrace{E(-kz)f'(z) - kE(-kz)f(z)}_{\frac{d}{dz}\left[E(-kz)f(z)\right]} \equiv 0$$

$$\frac{d}{dz}\left[E(-kz)f(z)\right] \equiv 0 \quad ?$$

Yes: $\frac{d}{dz}E(-kz) = E'(-kz)\frac{d}{dz}[-kz] = E(-kz)(-k)$

So $\boxed{E(-kz)f(z) \equiv C}$, a const.

Plug in $z=0$. See $C=f(0)$. $[E(0)=1]$

Stop the proof! Start over, but use $f(z)=E(kz)$.

Notice that $f'(z)=E'(kz)\frac{d}{dz}[kz] = kE(kz)$

$$= kf(z)$$

$f(z)=E(kz)$ satisfies hyp of lemma. Red box:

$$E(-kz)\,E(kz) = C = E(k\cdot 0) = 1$$

Aha! Just showed those two are non-zero and

$$E(-kz) = \frac{1}{E(kz)}.$$

Back to proof with original $f(z)$ =

Red box shows $f(z)=CE(kz)$

where $C=f(0)$. ✓

Very cool thing: Let $f(z)=E(z+w)$. [Think w=const].

$$f'(z)=E'(z+w)\frac{d}{dz}[z+w] = E(z+w)\cdot 1$$

$$= f(z) \leftarrow k=1$$

f satisfies hyp of lemma with $k=1$.

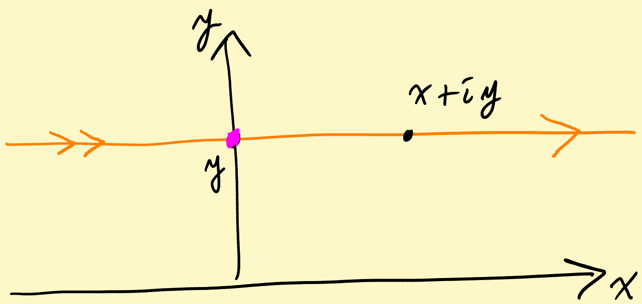
So $f(z)=CE(1\cdot z)$ where $C=f(0)$

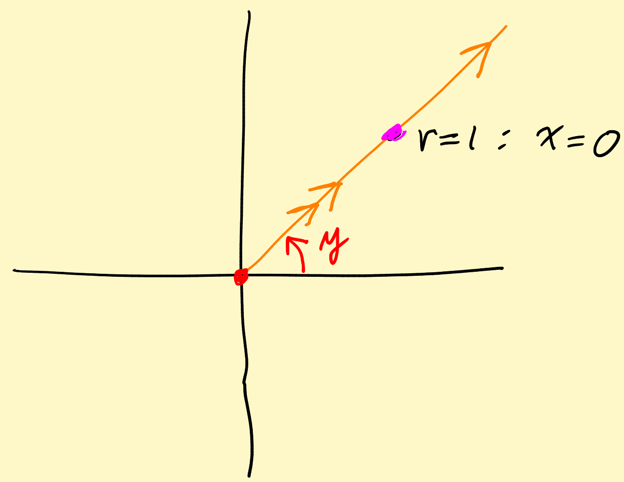
$$E(z+w) = \underbrace{E(0+w)}_{c}\, E(1\cdot z)$$

Aha! $E(z+w) = E(z)E(w)$ ✓

Wow! Can deduce truckloads of trig identities without drawing a single Δ!

<u>Graph of e^z</u>

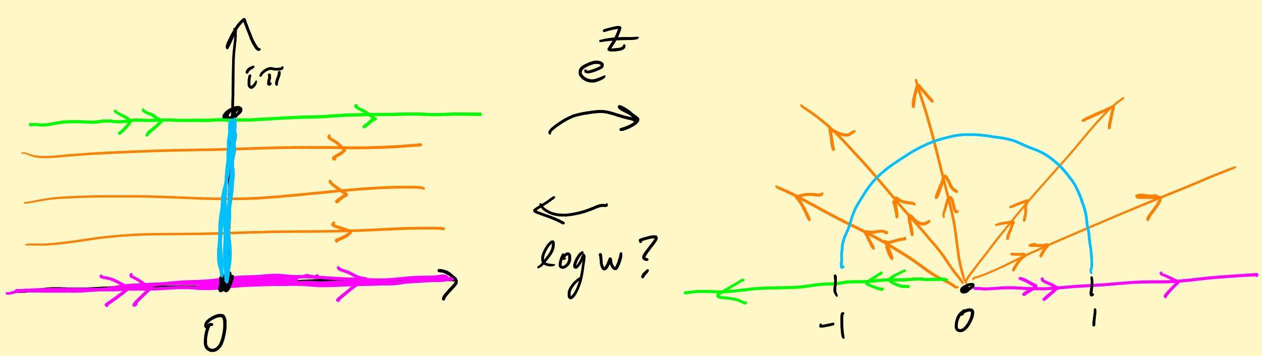




$$e^{x+iy} = \underbrace{e^x}_{r}\, e^{iy} \qquad y=\theta$$

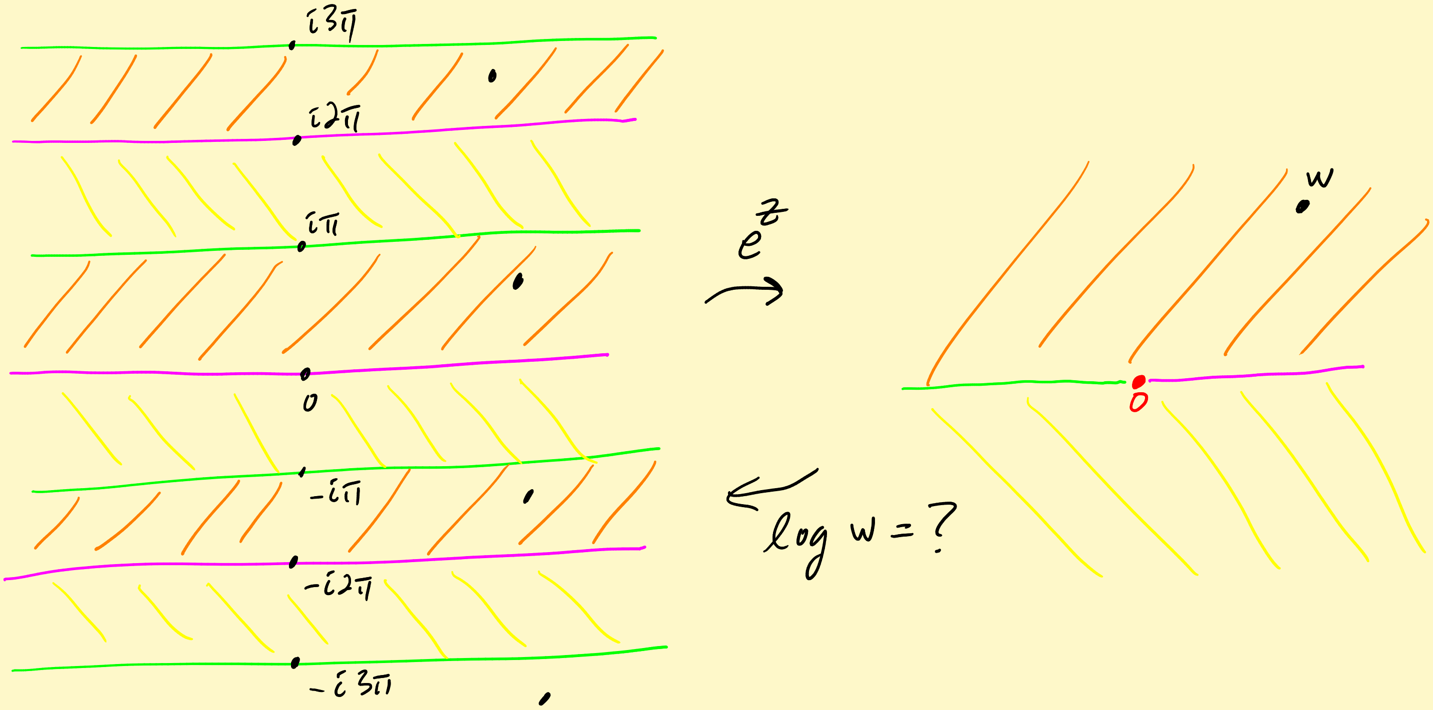
$$-\infty < x < \infty$$

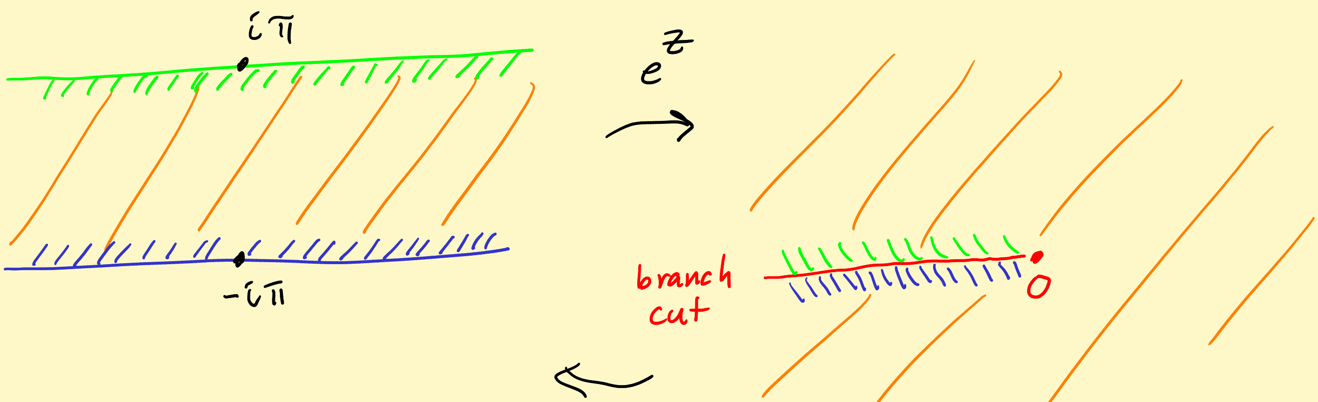
$$0 < e^x < \infty$$



<u>Fact</u>: e^z maps $\{z : 0 < \operatorname{Im} z < \pi\}$ (domain) 1-1 onto the Upper Half Plane (UHP)

$\{z : \operatorname{Im} z > 0\}$.





Log w

principal branch of the compex log fcn.

e^z maps $\{z : -\pi < \operatorname{Im} z < \pi\}$ 1-1 onto $\mathbb{C} - (-\infty, 0]$.

Log w is its <u>inverse</u>.

<u>Think</u> $w = e^z = \underbrace{e^x}_{r} \, \underbrace{e^{iy}}_{e^{i\theta}}$ where $r = |w| = e^x$. So $x = \operatorname{Ln}|w|$.

$$\begin{cases} x = \operatorname{Ln}|w| \\ y \in \arg w \end{cases} \leftarrow \text{set } \{\theta : w = re^{i\theta},\ r = |w|\}$$

$$z = \log w = (\operatorname{Ln}|w|) + i\theta, \quad \theta \in \arg w$$

My notations and def^{n}s

Ln is reserved for the real natural log on $\mathbb{R}^+$.

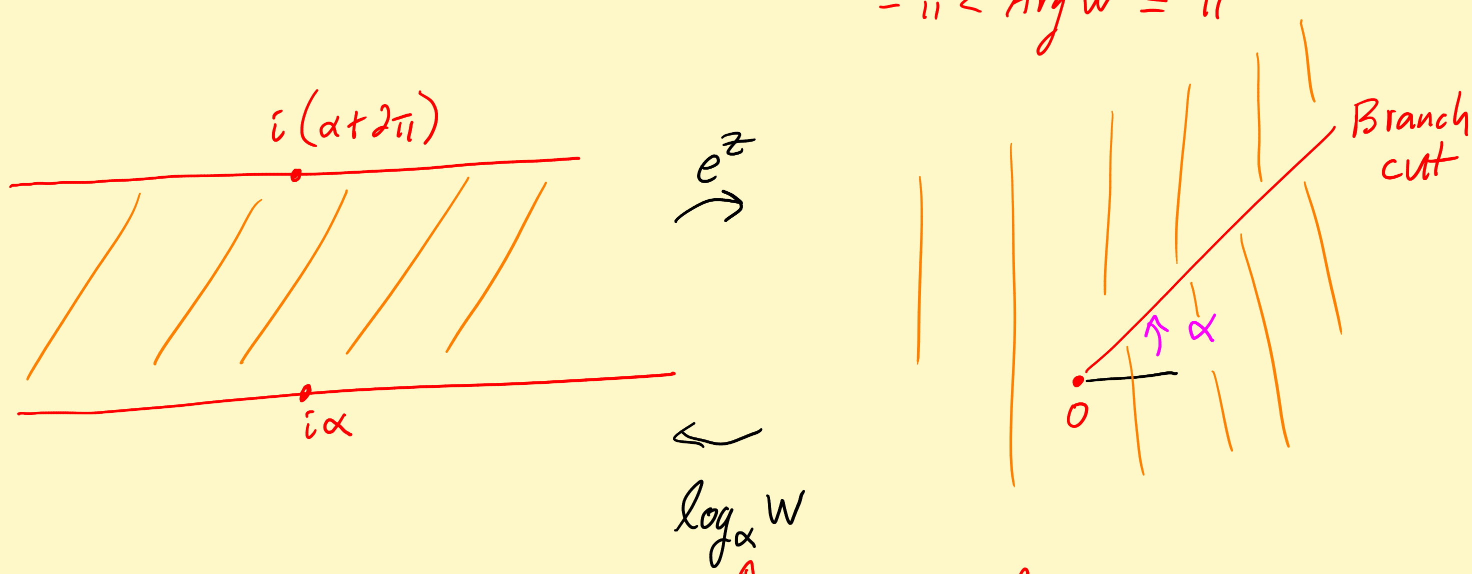
$$\log w = \{z : e^z = w\} = \{\operatorname{Ln}|w| + i\theta : \theta \in \arg w\}$$

$$= \{\operatorname{Ln}|w| + i\theta : \theta = \operatorname{Arg} w + n2\pi : n \in \mathbb{Z}\}$$

principal arg

$$\operatorname{Log} w = \text{Princ log} = \operatorname{Ln}|w| + i \operatorname{Arg} w$$

$$-\pi < \operatorname{Arg} w \leq \pi$$

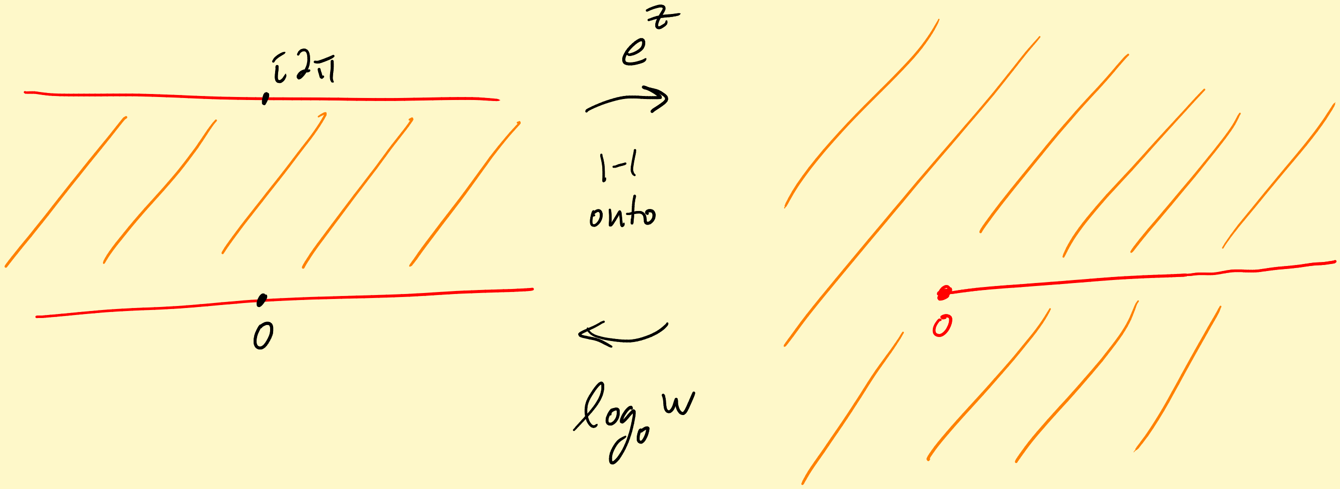


branch of log

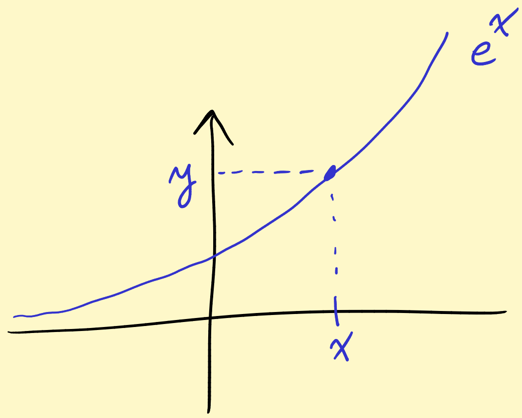
My defⁿ: $\log_\alpha w = \text{Ln}|w| + i\theta$ where $\theta \in \arg w$

$\alpha < \theta < \alpha + 2\pi$

Popular branch:



Real exponential



only one y with $e^x = y$.

$y = \text{Ln}\, x$

Ln only defined on $\mathbb{R}^+$

$$e^x : \mathbb{R} \xrightarrow[\text{onto}]{1\text{-}1} \mathbb{R}^+$$

$$\text{Ln}\, y : \mathbb{R}^+ \xrightarrow[\text{onto}]{1\text{-}1} \mathbb{R}$$