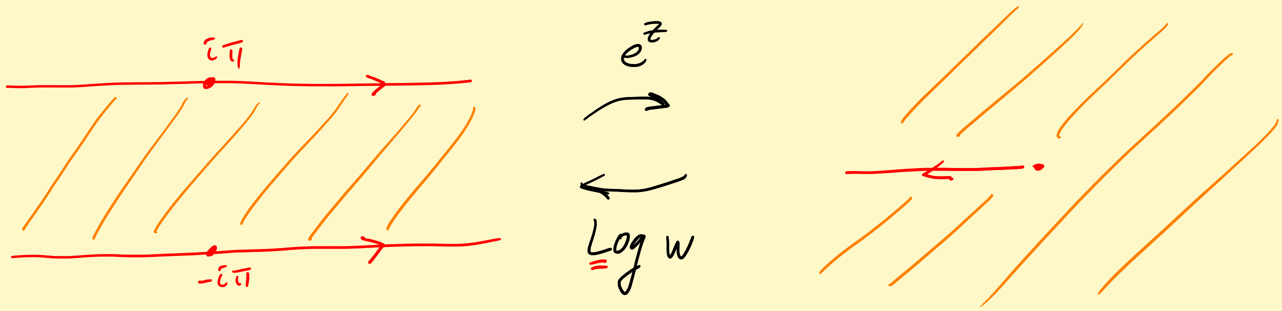
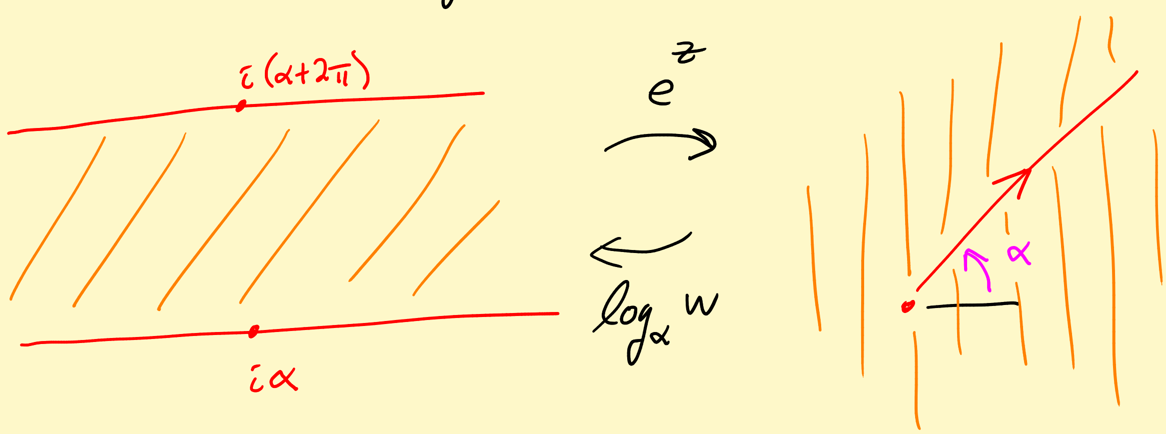
Lecture 9 Complex logarithm(s)

Read 3.1



Princ branch $\quad \underline{\underline{L}}og\, w = \operatorname{Ln}|w| + i\theta, \quad \theta \in \arg w : -\pi < \theta \le \pi$



$$\log_\alpha w = \operatorname{Ln}|w| + i\theta \qquad \theta \in \arg w, \ \alpha < \theta < \alpha + 2\pi$$

Cauchy-Riemann eqns in polar form

$$f(re^{i\theta}) = \bar{U}(r,\theta) + i\bar{V}(r,\theta)$$

If $\bar{U}$ and $\bar{V}$ are C^1-smooth on a polar rectange $\{(r,\theta) : 0 < r_1 < r < r_2, \ \alpha < \theta < \beta\}$ and

$$\begin{cases} \dfrac{\partial \bar{U}}{\partial r} = \dfrac{1}{r}\dfrac{\partial \bar{V}}{\partial \theta} \\[2ex] \dfrac{\partial \bar{V}}{\partial r} = -\dfrac{1}{r}\dfrac{\partial \bar{U}}{\partial \theta} \end{cases} \qquad \text{polar C-R eqns}$$

then f is analytic in polar rec.

EX $\quad \log_\alpha w = \ln|w| + i\theta$

$$\log_\alpha re^{i\theta} = \underbrace{\ln r}_{U} + i\underbrace{\theta}_{V} \qquad \begin{array}{l} r>0 \\ \alpha<\theta<\alpha+2\pi \end{array}$$

Polar C-R equns ✓ $\quad U, V$ C^1-smooth ✓

$\log_\alpha$ analytic ✓

Prob Find $\frac{d}{dz}(\log_\alpha z)$

Hmmm. $\quad (1) \quad e^{\log_\alpha z} = z$ always true

$(2) \quad \log_\alpha e^z = z + i2\pi n$ for some $n \in \mathbb{Z}$.

careful!

Check (1): $e^{\log_\alpha z} = e^{\ln|z| + i\theta} = |z| e^{i\theta} = z$ ✓

$\theta \in \arg z$.

Diff (1) using chain rule:

$$z = e^{\log_\alpha z} = E(\log_\alpha z)$$

$$\frac{d}{dz}: \qquad 1 = E'(\log_\alpha z)\, \frac{d}{dz} \log_\alpha z$$

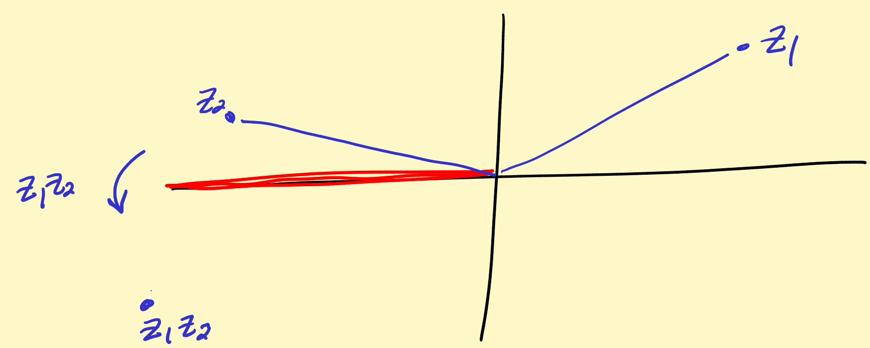
$$= \underbrace{e^{\log_\alpha z}}_{z} \frac{d}{dz} \log_\alpha z$$

Get $\boxed{\frac{d}{dz} \log_\alpha z = \frac{1}{z}}$

Warning! $\log_\alpha z_1 z_2 \overset{?}{=} \log_\alpha z_1 + \log_\alpha z_2$

Maybe not!

EX



$\text{Arg } z_j > 0$

$j = 1, 2$

$\text{Arg } z_1 z_2 < 0$

Fact: $\log_\alpha z_1 z_2 = \log_\alpha z_1 + \log_\alpha z_2 + i 2\pi n$

some $n \in \mathbb{Z}$

Book: $\log z_1 z_2 = \log z_1 + \log z_2$ as <u>sets</u>. True.

<u>Fact</u>: $\text{Log } z_1 z_2 = \text{Log } z_1 + \text{Log } z_2$ when z_1, z_2 in first quadrant

(or right half plane too)

Polar C-R equs: $f(x+iy) = u(x,y) + iv(x,y)$

$f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$

$$\begin{cases} \bar{U}(r,\theta) = u(r\cos\theta, r\sin\theta) \\ \bar{V}(r,\theta) = v(r\cos\theta, r\sin\theta) \end{cases}$$

Chain rule

$$\bar{U}_r = u_x \cos\theta + u_y \sin\theta$$

$$\bar{U}_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta)$$

$$\frac{1}{r}\bar{U}_\theta = u_x(-\sin\theta) + u_y\cos\theta$$

Similar for $\bar{V}$.

(*)
$$\underbrace{\begin{bmatrix} \bar{U}_r & \bar{V}_r \\ \frac{1}{r}\bar{U}_\theta & \frac{1}{r}\bar{V}_\theta \end{bmatrix}}_{\text{Polar C-R equ}} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{\text{has form}} \underbrace{\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}}_{\text{Cartesian C-R equs}}$$

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \qquad \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \qquad \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

Easy fact: Product of matrices in this form is in this form!

So: Cartesian C-R equs $\Rightarrow$ polar ✓.

Hmmm.
$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{\underbrace{\cos^2 + \sin^2}_{1}} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

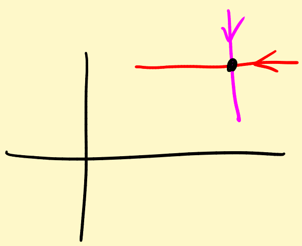
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of θ matrix is same form. Multiply on left by inverse.

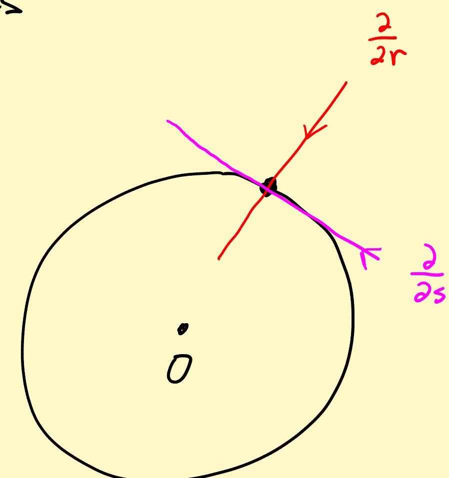
See: Polar eqns => cartesian ones. ✓

Or Start from first principles

Cartesian



Polar



equate DQs!

Fantastic thing about complex e^z and $\log z$.

In $\mathbb{R}$, entire zoo of elementary fcns.

$$e^x,\ \ln x,\ \sin x,\ \cos x,\ \tan x,\ \sinh x,\ \cosh^{-1} x$$
$$\sin^{-1} x,\ \sqrt{x},\ \sqrt[n]{x},\ \text{algebraic fcns.}$$

Liouville's thm $\int e^{-x^2}\,dx$ cannot be expressed in terms of elementary fcns!

Claim: zoo is only two animals! e^z, complex log.

EX: $z = \cos^{-1} w$

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$2w = e^{iz} + e^{-iz} \leftarrow \text{Mult by } e^{iz}$$

$$2w(e^{iz}) = (e^{iz})^2 + 1$$

$$0 = (e^{iz})^2 - 2w(e^{iz}) + 1 \leftarrow \text{quadratic equ!}$$

$$e^{iz} = \frac{2w + \sqrt{(-2w)^2 - 4}}{2} = w + \sqrt{w^2 - 1}$$

two complex $\sqrt{\ }$'s

$$iz = \log\left(w + \sqrt{w^2-1}\right)$$

$$z = \underbrace{-i \log\left(w + \sqrt{w^2-1}\right)}_{\cos^{-1} w}$$

Liouville: In $\mathbb{C}$: elementary fcns are "generated" by e^z and its inverse <u>and</u> the algebraic fcns.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \tan z = \frac{\sin z}{\cos z} \qquad \text{etc.}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$