Lecture 10 The complex inverse function thm, i^i, Sec 3.4

HWK3 due Thurs
Exam in class, Wed, March 5.

Complex inverse fcn thm If f is analytic on $D_r(a)$ and $f'(a) \neq 0$, then f^{-1} is well defined in a disc $D_\varepsilon(f(a))$ and f^{-1} is analytic there. And

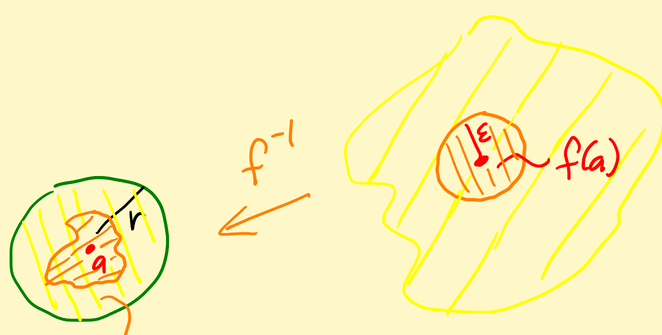
$$\boxed{\frac{d}{dw}\left(f^{-1}(w)\right) = \frac{1}{f'(f^{-1}(w))}}$$

MA 425 Define $f = u + iv$ analytic to mean u, v C^1-smooth and satisfy the C-R eqns. So $DQ = \dfrac{f(z)-f(a)}{z-a} \to f'(a)$.

MA 530 Analytic means $DQ \to f'(a)$ on an open set.

Remark Equivalent!

$\mathbb{C}$ Inv fcn thm



U open connected set — a domain

$$f : U \xrightarrow[\text{onto}]{1-1} D_\varepsilon(f(a)) \qquad f^{-1} \text{ is analytic on } D_\varepsilon(f(a)).$$

Follows from $\mathbb{R}^2 \to \mathbb{R}^2$ inv fcn thm: $f(x+iy) = u(x,y) + iv(x,y)$

$$(x, y) \longmapsto (u(x,y), v(x,y))$$

u, v C^1-smooth and satisfy C-R eqns.

Key Jacobian matrix $J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \underset{\text{C-R eqns}}{=} \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$

$$\det J_f = \det \underbrace{\begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}}_{\text{using C-R eqns}} = (u_x)^2 + (v_x)^2$$

$$= |\underbrace{u_x + i v_x}_{\substack{\text{Red box \#1} \\ \text{for } f'}}|^2 = |f'(z)|^2$$

$\neq 0$ at $z = a$. So hyp of $\mathbb{R}^2$ inv fcn thm holds.

U, V C^1-smooth

Get inverse fcn $F = U + iV$. $F = f^{-1}$

Hmmm $J_F = J_f^{-1} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^{-1}$

$$= \frac{1}{\underbrace{A^2 + B^2}_{\neq 0}} \underbrace{\begin{bmatrix} A & B \\ -B & A \end{bmatrix}}_{\text{See C-R eqns!}}$$

minuses on other diag

So f^{-1} analytic!

Formula is just the chain rule:

$$z = f^{-1}(f(z))$$

$$1 = (f^{-1})'(\underbrace{f(z)}_{w})\, f'(\underset{z=f^{-1}(w)}{z}) \checkmark$$

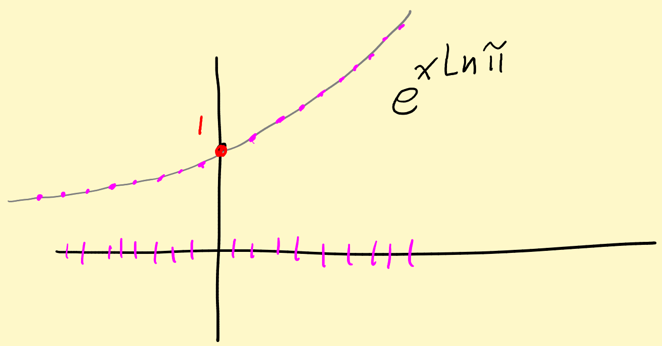
Remark Could have used this to see $\log w$ is analytic locally because $\frac{d}{dz}(e^z) = e^z \leftarrow$ never zero.

Problem What is i^i?

Hmmm. $\mathbb{R} \to \mathbb{R}$ $\qquad x^n = \underbrace{x \cdot x \cdots x}_{n\text{-times}}$

$$x^{p/q} = \sqrt[q]{x^p}$$

MA 161. What is $\pi^{\sqrt{2}}$?



Rational x-values. Dense.

$\pi^{p/q}$

$$\pi^{\sqrt{2}} = e^{\ln \pi^{\sqrt{2}}} = e^{\sqrt{2} \ln \pi}$$

Complex version: $\quad z^w = e^{w \log z} = \{ e^{w\zeta} : \zeta \in \log z \}$

∞ many values of $\log z$!

Def^n Principal value of

$$z^w = e^{w \operatorname{Log} z} \leftarrow \text{Princ branch}$$

EX Principal N-th root

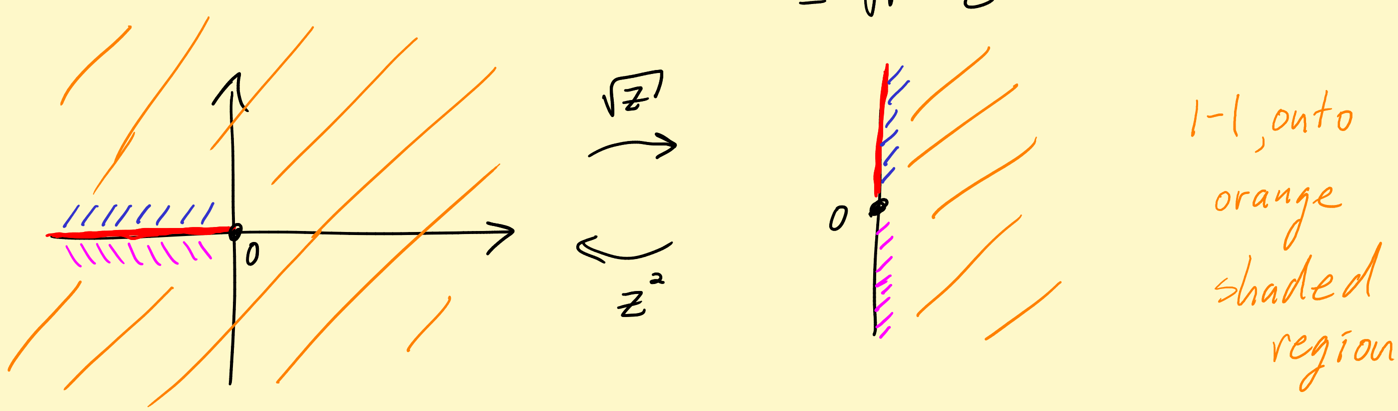
$$z^{1/N} = e^{\frac{1}{N}\operatorname{Log} z}$$

Princ square root $\quad z^{1/2} = e^{\frac{1}{2}\operatorname{Log} z}$

$$(re^{i\theta})^{1/2} = e^{\frac{1}{2}(\operatorname{Ln} r + i\theta)} \qquad -\pi < \theta \le \pi$$

$$= e^{\frac{1}{2}\operatorname{Ln} r}\, e^{i\theta/2}$$

$$= \sqrt{r}\, e^{i\theta/2}$$



Easy facts $\qquad z^n$ only has one value

$$z^n = e^{n \log z} = e^{n(\operatorname{Ln}|z| + i(\operatorname{Arg} z + 2n\pi))}$$

$$= e^{n \operatorname{Ln}|z|} \cdot e^{i n \operatorname{Arg} z} \underbrace{e^{i 2n\pi}}_{=1}$$

$$= \left(e^{\operatorname{Ln}|z|} e^{i \operatorname{Arg} z}\right)^n$$

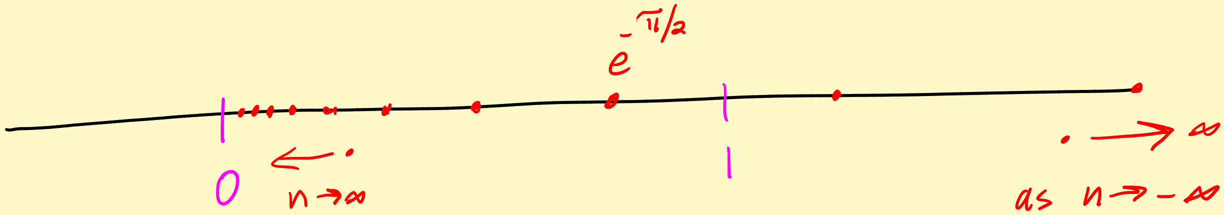
$$= \underbrace{z \cdot z \cdots z}_{n\text{-times}}$$

Also $z^{p/q}$ has q values in $\mathbb{C}$.

Princ value $i^i = e^{i \operatorname{Log} i} = e^{i(\underbrace{\operatorname{Ln}|i|}_{0} + i\frac{\pi}{2})} = e^{-\pi/2} \in \mathbb{R}!$

All values: $= e^{i \log i} = e^{i(\underbrace{\operatorname{Ln}|i|}_{0} + i(\frac{\pi}{2} + 2n\pi))}$

$$= e^{-(\frac{\pi}{2} + 2n\pi)}$$



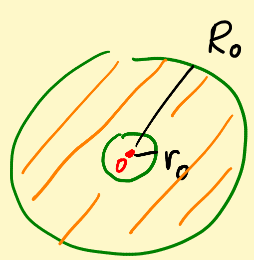
3.4 Washers, wedges, walls

Heat eqn: $\underbrace{\frac{\partial u}{\partial t}}_{} = \Delta u$

$\equiv 0$ at "steady state"

Harmonic fcns solve heat problems!

Washers $\underbrace{\operatorname{Ln}|z|}_{\operatorname{Ln} r} = \operatorname{Re}(\log z)$ is harmonic



Prob Find harmonic u with $u = \begin{cases} T_1 & \text{on } C_{R_0}(0) \\ T_2 & \text{on } C_{r_0}(0) \end{cases}$

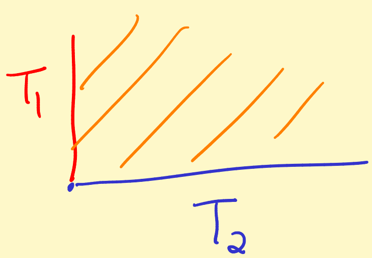
circles

Sol^{n}s $\quad \operatorname{Ln} r, \quad A \operatorname{Ln} r, \quad \underbrace{A \operatorname{Ln} r + B}_{\text{works!}}$

$$\begin{cases} A \operatorname{Ln} r_0 + B = T_2 \\ A \operatorname{Ln} R_0 + B = T_1 \end{cases} \qquad \text{get } A, B.$$

Sol^n $\quad A \operatorname{Ln}|z| + B$.

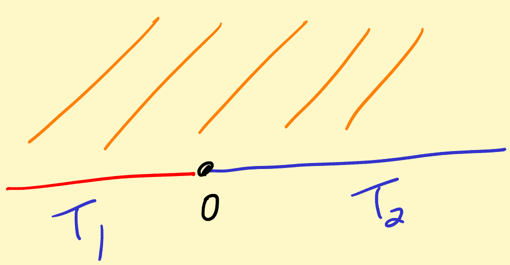
Wedges, walls. $\quad u = \operatorname{Arg} z = \operatorname{Im}(\operatorname{Log} z)$ is harmonic!



Sol^{n}s

$\operatorname{Arg} z, \quad A \cdot \operatorname{Arg} z, \quad \underline{A \cdot \operatorname{Arg} z + B}$

works!



same idea works.