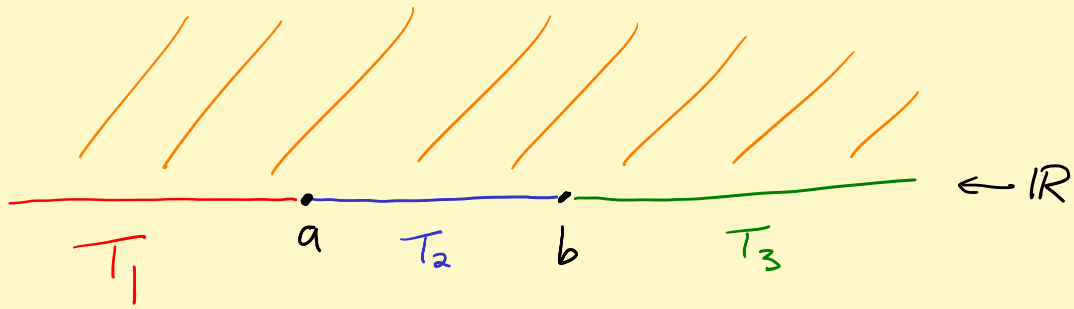
Lecture 11 Complex integration (Start Chap 4. HWK 4 due next Thurs)

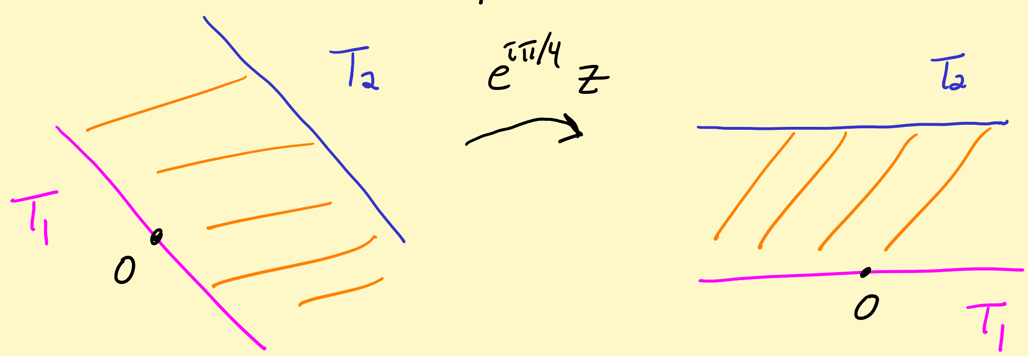
3.4 hints



$$u = A \cdot \operatorname{Arg}(z-a) + B \cdot \operatorname{Arg}(z-b) + C$$

π to left of
a on $\mathbb{R}$,
0 to right,
$\Delta \equiv 0$ on UHP

Just make it work: Get 3 equs in 3 unknowns. ✓



Sol^n

$u(e^{i\pi/4} z)$

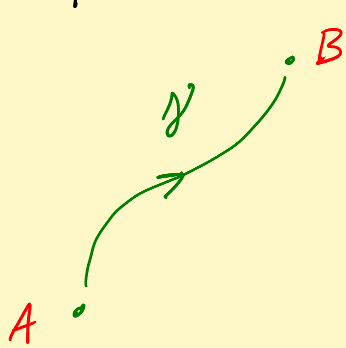
$\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$A \operatorname{Im} z + B$

$u(z)$

Fact: harmonic fcns composed with analytic fcns are harmonic

Curves in $\mathbb{C} \approx \mathbb{R}^2$



$\gamma : (x(t), y(t))$

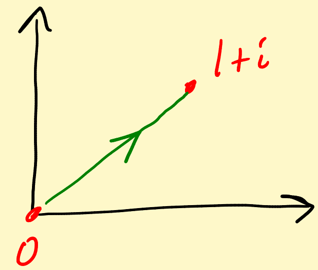
$a \le t \le b$

(in $\mathbb{R}^2$)

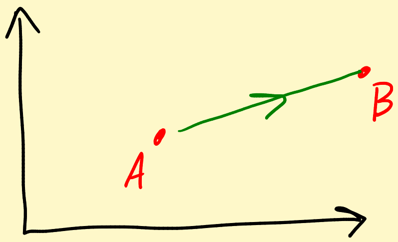
In $\mathbb{C}$: $z(t) = x(t) + i\,y(t)$, $a \le t \le b$

$z : [a,b] \to \mathbb{C}$

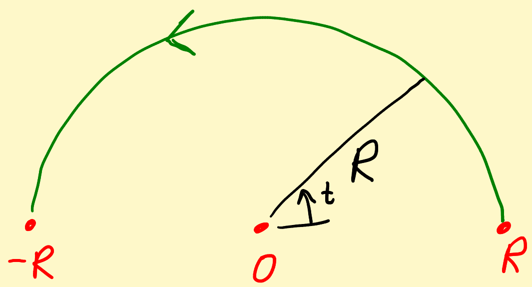
<u>EX</u>



$L: \; z(t) = t(1+i), \; 0 \le t \le 1$



$L_A^B : A + t(B-A), \; 0 \le t \le 1$



$C_R : \; z(t) = Re^{it}, \; 0 \le t \le \pi$

<u>Two kinds of fcns</u> 1) $f : \Omega \to \mathbb{C}$ <u>analytic</u> fcns

($f = u + iv$, u, v C^1-smooth, satisfy C-R equs
$\Rightarrow$ DQ $\to f'$ and u, v C^∞-smooth)

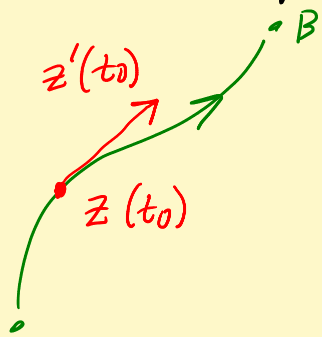
2) $z(t)$ $\quad z : [a,b] \to \mathbb{C}$, $z(t) = x(t) + i\,y(t)$

Define $z'(t_0) = \lim\limits_{t \to t_0} \dfrac{z(t) - z(t_0)}{t - t_0}$

$$= \lim_{t \to t_0} \left[\frac{x(t) - x(t_0)}{t - t_0} + i \, \frac{y(t) - y(t_0)}{t - t_0} \right]$$

$$= x'(t_0) + i \, y'(t_0)$$

Remark $z'(t)$ represents the "velocity vector" as a complex number.



In pic $z'(t_0)$ is a complex # that I place in plane thinking of $z(t_0)$ as origin

Two kinds of integrals

Basic kind 1) $\int_a^b z(t)\, dt$

$$= \lim_{\Delta t \to 0} \underbrace{\sum_{n=1}^{N} z(t_n)\, \Delta t}_{\left(\sum_1^N x(t_n)\Delta t\right) + i \left(\sum_1^N y(t_n)\Delta t\right)}$$

$$= \int_a^b x(t)\, dt + i \int_a^b y(t)\, dt \leftarrow \text{Def}^{n}$$

Basic inequality #1: $\left| \int_a^b z(t)\, dt \right| \le \left(\max_{a \le t \le b} |z(t)| \right) (b - a)$

Why $\left|\int_a^b z(t)\,dt\right| = \left|\lim_{\Delta t\to 0} \sum_1^N z(t_n)\,\Delta t\right|$

$$\le \lim_{\Delta t \to 0} \sum_1^N |z(t_n)|\,\Delta t$$

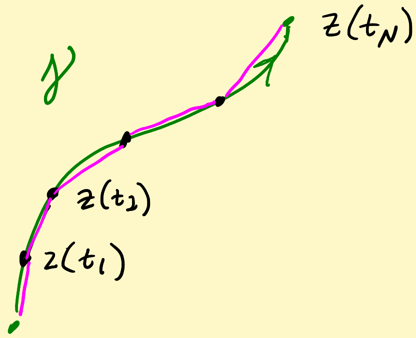
"="

$$\le \int_a^b |z(t)|\,dt \le \left(\max_{a\le t\le b} |z(t)|\right)\cdot(b-a) \checkmark$$

Baby lemma: $\int_a^b K\,z(t)\,dt = K\int_a^b z(t)\,dt, \quad K\in\mathbb{C}$

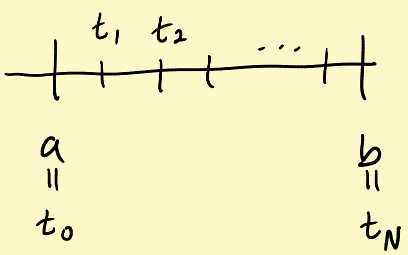
Pf Write out both sides. See $=$ $\checkmark$

Integrate $f(z)$ along γ (Konrad Knopf Dover book)



$$\int_\gamma f\,dz$$

$$= \lim_{\Delta t\to 0} \sum_{n=1}^N f(z(t_n))\underbrace{(z(t_n)-z(t_{n-1}))}_{\Delta z_n}$$



Hmmm $\Delta z_n = \dfrac{z(t_n)-z(t_{n-1})}{\Delta t}\cdot \Delta t$

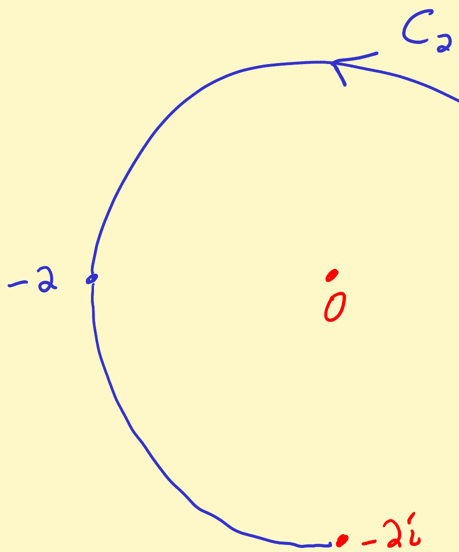
$$\approx z'(t_{n-1})\cdot\Delta t$$

when Δt is small

$$\Delta z \approx z'(t)\,\Delta t$$

Def^n $$\int_\gamma f\, dz = \text{Lim} = \int_a^b f(z(t))\, \underbrace{z'(t)\, dt}_{\text{Think } dz = z'(t)dt}$$

EX



C_2: $z(t) = 2e^{it}$ $\quad 0 \le t \le \frac{3\pi}{2}$

$= 2\cos t + i\, 2\sin t$

$$\int_{C_2} e^z\, dz = \int_0^{3\pi/2} \underbrace{e^{(2\cos t + i 2\sin t)}}_{e^{2\cos t}\left(\cos(2\sin t) + i \sin(2\sin t)\right)} \left[-2\sin t + i\, 2\cos t\right] dt$$

Multiply out. Take real and imag parts. Ugh!

Thankfully, Fund Thm of Calc for analytic fcns

$$\int_{\gamma_A^B} e^{kz}\, dz = \int_{\gamma_A^B} \frac{d}{dz}\left[\frac{1}{k} e^{kz}\right] dz = \frac{1}{k} e^{kz} \Big|_A^B$$

Two chain rules 1) f, g analytic. Then

$h(z) = f(g(z))$ analytic and $h'(z) = f'(g(z))\, g'(z)$.

2) f analytic, $z(t) : [a,b] \to \Omega \subset \mathbb{C}$.

$$\frac{d}{dt}\left[f(z(t)) \right] = \underbrace{f'(z(t))}_{\text{complex deriv}} \underbrace{z'(t)}_{\text{new deriv}}$$

Why 2: $z(t) = x(t) + i y(t)$

$f(x+iy) = u(x,y) + i v(x,y)$

$$\frac{d}{dt}\left[f(z(t)) \right] = \frac{d}{dt}\left[u(x(t), y(t)) + i v(x(t), y(t)) \right]$$

$$= \left(u_x \dot{x} + \underset{\substack{\| \\ -v_x}}{u_y} \dot{y} \right) + i \left(v_x \dot{x} + \underset{\substack{\| \\ u_x}}{v_y} \dot{y} \right)$$

C-R eqns

$$= \Big[\underbrace{u_x + i v_x}_{\text{Red box \#1}} \Big] \cdot \Big[\underbrace{\dot{x} + i \dot{y}}_{z'(t)} \Big]$$

$$= f'(z(t)) \cdot z'(t) \quad \checkmark$$

Next time: 2 Fund Thm Calculus:

$$\begin{cases} \int_a^b z'(t)\, dt = z(b) - z(a) \\ \int_{\gamma_A^B} f'(z)\, dz = f(B) - f(A) \end{cases}$$