Lesson 12 Complex integration

Two kinds of functions : $w = f(z)$ $\quad f : \Omega \to \mathbb{C}$

$$z(t) = x(t) + iy(t) \qquad z : [a,b] \to \mathbb{C}$$

Two kinds of derivatives: $f'(z)$ complex deriv

$$z'(t) = x'(t) + iy'(t)$$

Two kinds of integrals : $\int_a^b z(t)\,dt = \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,\underbrace{z'(t)\,dt}_{dz}$$

Two kinds of chain rules: $h(z) = f(g(z))$. $\quad h'(z) = f'(g(z))g'(z)$

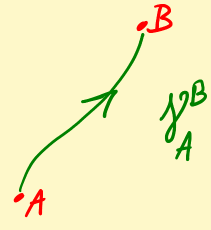
$$\frac{d}{dt} f(z(t)) = f'(z(t))z'(t)$$

$\mathbb{C}$-derivative

Two Fundamental theorems of Calculus :

1) $\displaystyle\int_a^b z'(t)\,dt = z(b) - z(a)$

2) $\displaystyle\int_{\gamma_A^B} f'(z)\,dz = f(B) - f(A)$



Two Basic inequalities : 1) $\left|\int_a^b z(t)\,dt\right| \le M\cdot(b-a)$

$M = \max_{t\in[a,b]} |z(t)|$

2) $$\left|\int_\gamma f(z)\,dz\right| \le \left(\max_{z\in\gamma} |f(z)|\right) \text{Length}(\gamma)$$

Fund Thm Calc #1

$$\int_a^b z'(t)\,dt = \underbrace{\int_a^b x'(t)\,dt}_{=x(b)-x(a)} + i\underbrace{\int_a^b y'(t)\,dt}_{=y(b)-y(a)}$$

$$= z(b) - z(a) \checkmark$$

Fund Thm Calc #2

$$\int_{\gamma_A^B} f'(z)\,dz = \int_a^b \underbrace{f'(z(t))\,z'(t)}_{\frac{d}{dt}f(z(t))}\,dt$$

$$= f(\underbrace{z(b)}_{B}) - f(\underbrace{z(a)}_{A}) \checkmark$$

Basic est #1. Last time ✓

Basic est #2

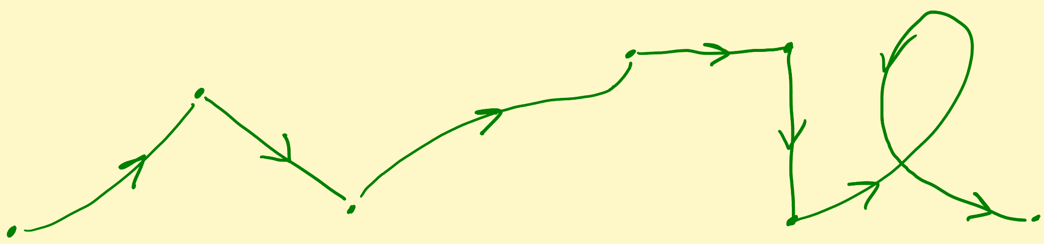
$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$$

$$\leq \int_a^b \underbrace{|f(z(t))|}_{\leq M} \cdot \underbrace{|z'(t)|}_{\sqrt{(\dot{x})^2+(\dot{y})^2}}\,dt$$

$$\leq M \underbrace{\int_a^b \underbrace{\sqrt{(x'(t))^2+(y'(t))^2}\,dt}_{ds}}_{\text{Length}(\gamma)}$$

Where $M = \max\limits_{a\leq t\leq b} |f(z(t))| = \max\limits_{\gamma} |f|$

Important I've been assuming γ is $\mathcal{C}^1$-smooth: $x(t), y(t), x'(t), y'(t)$ exist and are continuous on $[a,b]$. Easy to generalize facts to piecewise $\mathcal{C}^1$-smooth curves:



Continuous, but finitely many "corners" are allowed.

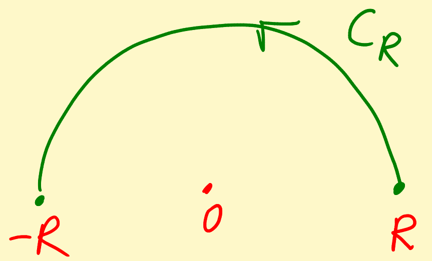
Basic facts hold for piecewise $\mathcal{C}^1$-smooth curves.

EX $$\int_\gamma f'\,dz = \sum_1^N \int_{\gamma_n} f'\,dz$$

$$= \left[f(z_2) - f(z_1)\right] + \left[f(z_3) - f(z_2)\right] + \left[f(z_4) - f(z_3)\right] + \cdots \left[f(z_N) - f(z_{N-1})\right]$$

$$= f(z_N) - f(z_1) \quad \checkmark$$

EX



$C_R: z(t) = Re^{it},\ 0 \le t \le \pi$

$z(t) = R\,E(it)$

$z'(t) = R\,E'(it)\,\frac{d}{dt}\left[it\right]$

$= R\,E(it)\cdot(i)$

$$\boxed{z'(t) = iRe^{it}}$$

EX $\displaystyle\int_{C_R} e^{2z}\,dz = \int_0^{2\pi} e^{2Re^{it}}\, iRe^{it}\,dt$ ← Ugh!

$$\underset{\substack{\uparrow\\ \text{better}}}{=} \int_{C_R} \frac{d}{dz}\left[\tfrac{1}{2}e^{2z}\right] dz$$

$$= \left[\tfrac{1}{2}e^{2z}\right]_R^{-R} = \tfrac{1}{2}e^{-2R} - \tfrac{1}{2}e^{2R}$$

$$= -\sinh 2R$$

Coming soon: Cauchy's theorem: f analytic on a domain Ω with no "holes" (a simply connected domain), then $\int_\gamma f\,dz = 0$ for all closed curves in Ω.

Funny thing: Reverse is true too: If $f: \Omega \to \mathbb{C}$ is continuous on a domain Ω and $\int_\gamma f\,dz = 0$ for all closed curves in Ω, then f is analytic.

Must study $\int_\gamma f\,dz$!

Book 4.4b : Green's thm approach to Cauchy's Thm

Skip 4.4a for now — Homotopy approach.

Big fact Suppose f is analytic on a domain Ω.
Then f has an analytic antiderivative $F(z)$
(meaning that $F'(z) = f(z)$ for $z \in \Omega$)
if and only if $\int_\gamma f\,dz = 0$ for all closed curves γ in Ω.

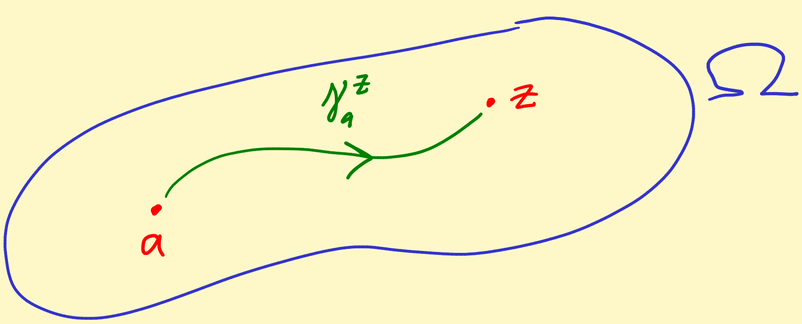
Pf ($\Rightarrow$) easy. Assume $F'(z) = f(z)$. γ closed.

$$\int_\gamma f\,dz = \int_\gamma F'\,dz = F(\text{END}) - F(\text{START}) = 0 \checkmark$$

same pt

($\Leftarrow$) We assume $\int_\gamma f\,dz = 0$ for all closed γ in Ω.

[Hmmm. If I had F with $F' = f$ . . .



a fixed.
Slide z around.

$$\int_{\gamma_a^z} \underbrace{F'(w)}_{f(w)}\,dw = F(z) - \underbrace{F(a)}_{\text{constant!}}$$

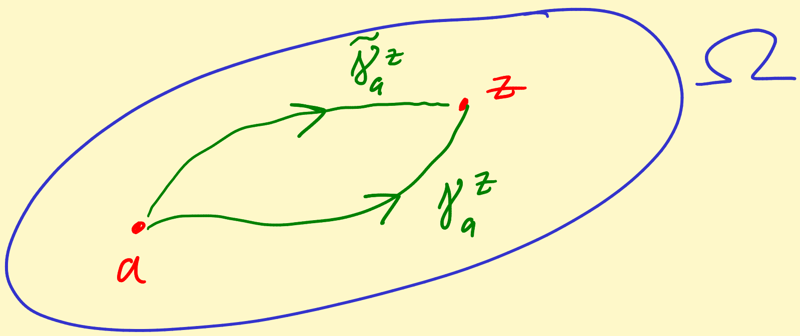
Hmmm. $\underbrace{F'(z)}_{f(z)} = \frac{d}{dz}\left[\underbrace{\int_{\gamma_a^z} f(w)\,dw}\right]$]

This is what we want!

<u>Big idea</u>. "Define" $F(z) = \int_{\gamma_a^z} f(w)\,dw$.

<u>Step 1</u> Show F is <u>well</u> defined.

Easy



Aha! Let $\Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$ ← is a <u>closed</u> curve

$$0 \underset{\text{hyp}}{=} \int_\Gamma f\,dw = \int_{\gamma_a^z} f\,dw + \underbrace{\int_{-\tilde{\gamma}_a^z} f\,dw}_{= -\int_{\tilde{\gamma}_a^z} f\,dw}$$

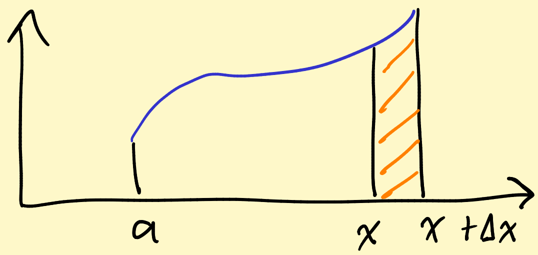
easy fact from def^{n}s.

So $\int_{\gamma_a^z} f\,dw = \int_{\tilde{\gamma}_a^z} f\,dw$ ← <u>well</u> defined ✓

Step 2 Show $F'(z) = f(z)$.

Next time. Feel like freshman calc

$$\frac{d}{dx}\left[\int_a^x h(t)\,dt\right] = h(x)$$



$\frac{\text{Area}}{\Delta x} \approx \text{height} \approx h(x)$