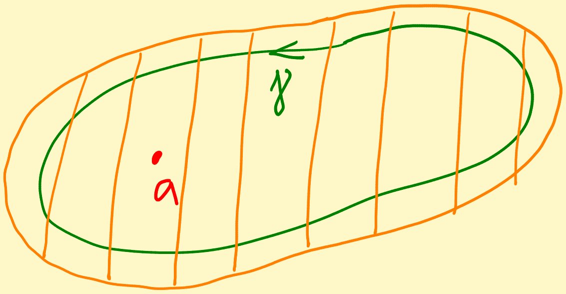
# Lecture 15 Liouville's theorem & the Fundamental theorem of Algebra




$f$ analytic inside and on counterclockwise simple closed curve $\gamma$, $a \in$ inside of $\gamma$.

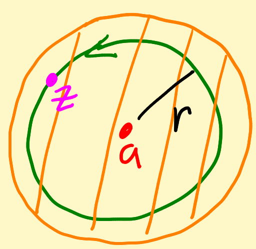
$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$

$$f'(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^2}\, dz$$

$$\vdots$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

## Cauchy estimates




$C_r(a)$

$$|f^{(n)}(a)| \le \frac{n!\, M}{r^n} \quad \text{where} \quad M = \max_{C_r(a)} |f|.$$

Pf:
$$|f^{(n)}(a)| = \left| \frac{n!}{2\pi i} \int_{C_r(a)} \frac{f(z)}{\underbrace{(z-a)}_{|z-a|=r}{}^{n+1}}\, dz \right|$$

$$\le \frac{n!}{2\pi} \left( \max_{C_r(a)} |f| \right) \frac{1}{r^{n+1}} \cdot \underbrace{(2\pi r)}_{\text{Length}(C_r(a))}$$

$$\leq \frac{n!M}{r^n}$$

Liouville's thm $\underbrace{\text{Bounded}}_{|f(z)|<M}$ $\underbrace{\text{entire}}_{f \text{ analytic on } \mathbb{C}}$ fcns must be const.

Pf: $n=1$ Cauchy est. Pick any pt $a \in \mathbb{C}$. Assume $|f(z)| < M$ on $\mathbb{C}$. Now $|f'(a)| \leq \frac{1!M}{r^1}$.

Aha! Can let $r \nearrow \infty$. See $f'(a) = 0$.

But $a$ is arbitrary. So $f' \equiv 0$ on $\mathbb{C}$ and so $f \equiv$ const on $\mathbb{C}$.

Next: Liouville's $\Rightarrow$ Fund Thm Alg.

Basic polynomial estimate $P(z) = a_N z^N + \cdots + a_1 z + a_0$ where $a_N \neq 0$ and $N \geq 1$. $\Leftarrow$ P not constant.

There are real constants $0 < A < B$ and a radius $R > 0$ such that

$$A|z|^N < |P(z)| < B|z|^N$$

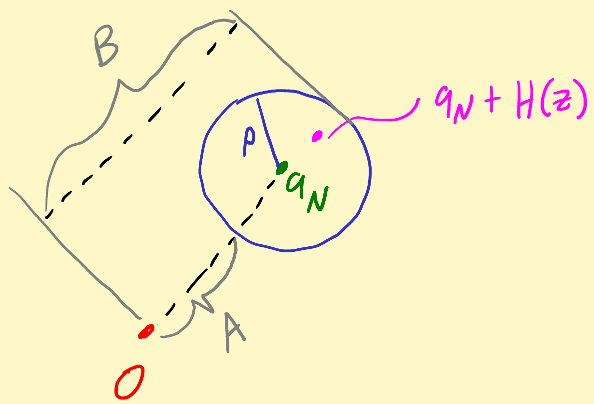
when $|z| > R$.

Think: $|P(z)|$ blows up (grows) like $|z|^N$ as $|z| \to \infty$.

**Remark** Note all the zeroes of $P(z)$ fall inside $D_R(0)$.

**Pf of poly est** $$P(z) = z^N \left[ a_N + \underbrace{\frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N}} \right]$$

$H(z) \to 0$ as $z \to \infty$




See $a_N + H(z)$ is between circles of radii $A$ & $B$

when $|H(z)| < \rho < |a_N|$.

$A < \left|\frac{P(z)}{z^N}\right| < B$ when $|H(z)| < \rho$, which

happens when $|z| > R$ for some $R$.

**Fun fact** By taking $\rho$ small, get $0 < A < |a_N| < B$

and $A$ and $B$ can as close to $|a_N|$ as desired.

Books: Take $\rho = \frac{|a_N|}{2}$.

**Fund thm Alg** A complex poly of deg $N \geq 1$ has a root in $\mathbb{C}$. [ $r$ is a root means $P(r) = 0$ ]

**Pf:** Suppose we have a poly $P(z)$ of deg $N \geq 1$ that does not have a root in $\mathbb{C}$. Then $\frac{1}{P(z)}$ is entire! Basic poly est:

$$\underline{A|z|^N < |P(z)|} < B|z|^N \ ; \ |z| > R.$$

$$\left|\frac{1}{P(z)}\right| < \frac{1}{A|z|^N} \quad \text{when } |z| > R.$$

[Can take $R > 1$ so that $|z|^N > R^N$ when $|z| > R$.]

Get $\left|\frac{1}{P(z)}\right| < \frac{1}{AR^N}$ when $|z| > \mathrm{Max}(R, 1)$.

<u>And</u> $\frac{1}{P(z)}$ analytic $\Rightarrow$ $\frac{1}{P(z)}$ continuous

$\Rightarrow$ $\left|\frac{1}{P(z)}\right|$ continuous, <u>and</u> a continuous

fcn on a <u>closed and bounded</u> (compact) set

like $\overline{D_R(0)} = \{z : |z| \le R\}$ <u>attains</u> its max.

Get bound for $\frac{1}{P(z)}$ given by $\mathrm{Max}\left(\frac{1}{AR^N}, \max_{\overline{D_R(0)}}\left|\frac{1}{P}\right|\right)$.

Liouville's $\Rightarrow$ $\frac{1}{P(z)} \equiv$ const. Constant <u>not</u> zero.

[Pick $z=0$ to see.] So $P(z) \equiv$ const. Crazy. ↯

$\left[\ \frac{d^N}{dz^N} P(z) = N!\, a_N \neq 0\ \right]$ ← or use poly est.

Cor Polys can be factored over $\mathbb{C}$.

Pf Euclidean algorithm

$$P(z)\overline{\big)\,Q(z)}^{\;q(z)}$$

$$\vdots$$

$$\underbrace{r(z)}_{\deg < \deg P} \leftarrow \text{remainder}$$

$$Q = qP + r$$

Say $a$ is a root of $Q(z)$.

$$(z-a)\overline{\big)\,Q(z)}^{\;q(z)}$$

$$\vdots$$

$b \leftarrow$ constant. $\deg r < \deg(z-a) = 1$.

$$Q(z) = (z-a)\,q(z) + b.$$

Plug $z=a$: $0 = \underbrace{Q(a)}_{0} = \underbrace{(a-a)}_{0}\,q(a) + b$. So $b=0$.

Fact: If $Q(a)=0$, $z-a$ divides $Q(z)$.

Note: $\deg q(z) = \deg Q - 1$.

Use Fund thm alg repeatedly. Get

$$P(z) = (z-r_1)(z-r_2)(z-r_3)\cdots(z-r_N)\,b$$

$\uparrow$ last quotient $\deg q = 0$.

Easy to see $b = a_N$.

Ok for some r's to repeat.

$$P(z) = a_N (z - r_1)^{m_1} (z - r_2)^{m_2} \cdots (z - r_\ell)^{m_\ell}$$

where $m_1 + m_2 + \cdots + m_\ell = N$