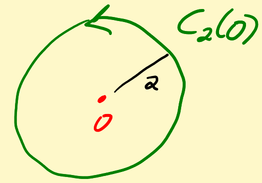
Lecture 16 Morera's theorem, Maximum modulus principle

Problem: $C_2(0) = 2e^{it}, \quad 0 \le t \le 2\pi$



Compute $$I = \int_{C_2(0)} \frac{e^{2z}}{(z-b)^3(z-a)}\, dz, \quad a, b \notin C_2(0)$$

<u>Case</u> a, b both outside $C_2(0)$. $I = 0$ by Cauchy's thm.

<u>Case</u> a inside, b outside. Hmmm. Let $f(z) = \dfrac{e^{2z}}{(z-b)^3}$.

It is analytic inside and on $C_2(0)$. Cauchy integral formula:
$$f(a) = \frac{1}{2\pi i} \int_{C_2(0)} \frac{f(z)}{z-a}\, dz = \frac{1}{2\pi i} I.$$

So $$I = 2\pi i\, f(a) = \underline{2\pi i \frac{e^{2a}}{(a-b)^3}}$$

<u>Case</u> b inside, a outside. This time, let $f(z) = \dfrac{e^{2z}}{z-a}$

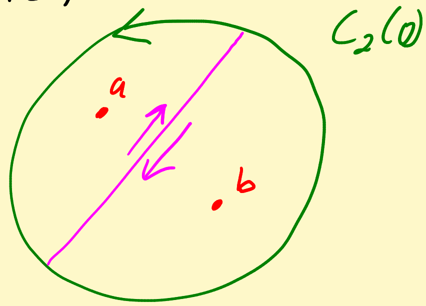
$$f^{(n)}(b) = \frac{n!}{2\pi i} \int_{C_2(0)} \frac{f(z)}{(z-b)^{n+1}}\, dz$$

want $n+1 = 3$.
$n = 2$

$$\boxed{f''(b) = \frac{2!}{2\pi i} \underline{I}}$$

So $I = \pi i \dfrac{d^2}{dz^2}\left[\dfrac{e^{2z}}{z-a}\right]\Bigg|_{z=b}$

<u>Case</u> a, b both inside. $(a \neq b)$



Aha! Add up 2 $\int$'s to get $\int_{C_2(0)}$ and use previous cases!

<u>Case</u> $a = b$ inside

$$I = \int_{C_2(0)} \frac{e^{2z}}{(z-a)^4}\, dz$$

$n+1 = 4$
$n = 3$

Use $f(z) = e^{2z}$. $\quad f'''(a) = \frac{3!}{2\pi i} I$.

<u>OR</u> Partial fractions:

$$\frac{1}{(z-b)^3(z-a)} = \underbrace{\frac{A}{(z-b)^3} + \frac{B}{(z-b)^2} + \frac{C}{z-b}} + \frac{D}{z-a}$$

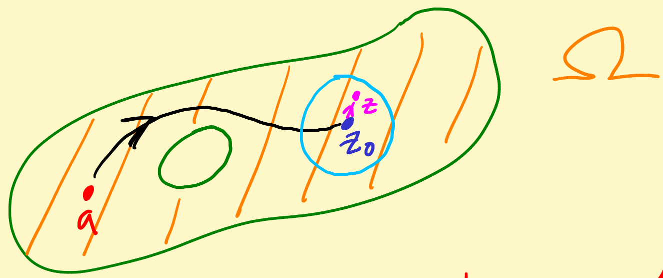
Know big fact now: f analytic $\Rightarrow f'$ analytic $\Rightarrow f''$ analytic $\Rightarrow$

$$\begin{array}{ccc} \Downarrow & \Downarrow & \Downarrow \\ f \text{ cont} & f' \text{ cont} & f'' \text{ cont} \end{array}$$

<u>Morera's Theorem</u> Given a <u>continuous</u> complex valued fcn f on a domain Ω. If $\int_\gamma f\, dz = 0$ for every closed curve γ in Ω, then f is <u>analytic</u>.

Remark Reverse is true when Ω is simply connected.

Pf of Morera's (Sneaky!)



$$F(z) = \int_{\underbrace{\gamma_a^z}_{\gamma_a^{z_0} \cup L_{z_0}^z}} f(w)\,dw$$

$\leftarrow$ hyp is what we need to see $F(z)$ is well defined.

Now, the proof we did that $F'(z) = f(z)$ goes through without issue. [This is where we need to use continuity of f.]

Last step Hmmm $F' = f$. So F is analytic.

But F analytic $\Rightarrow$ $F' = f$ analytic. ✓

Fantastic thing We will use Morera's to show that power series converge to analytic fcns without ever writing down a diff quotient!

$$\int_\gamma \left[\sum_{n=0}^{\infty} a_n (z-z_0)^n\right] dz = \sum_{n=0}^{\infty} a_n \left[\int_\gamma (z-z_0)^n\, dz\right]$$

we will know that p.s. converge uniformly (to a continuous fcn)

$$\underbrace{\int_\gamma \frac{d}{dz}\left[\frac{1}{n+1}(z-z_0)^{n+1}\right]dz}_{=0}$$

when γ closed

Maximum modulus thm

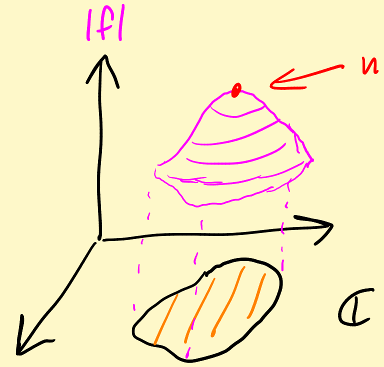
Suppose Ω is a bounded domain (meaning that $\Omega \subset D_R(0)$ for some $R>0$), f continuous on $\overline{\Omega}$ ($\overline{\Omega}$ = closure of Ω), f analytic in Ω.

Then $|f|$ attains its maximum value in $\overline{\Omega}$ on the boundary of Ω.

Fancier version: If $|f|$ has a local max at a pt inside Ω, then $f \equiv$ constant on Ω.

Note: Fancier version holds even when Ω not bounded.

Feeling:



no bump to sit on when f analytic

Key: Analytic fcns satisfy the averaging property

f analytic on Ω, $\overline{D_r(a)} \subset \Omega$.

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\,d\theta \leftarrow \text{ave of } f \text{ on } C_r(a)$$

$$= \frac{1}{\underbrace{2\pi r}_{\text{length } C_r(a)}}\int_0^{2\pi} f(a + re^{i\theta})\,\underbrace{r\,d\theta}_{ds}$$

$s=$ arc length

Why: $$f(a) = \frac{1}{2\pi i}\int_{C_r(a)} \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{\underbrace{(a+re^{i\theta})}_{z(\theta)}-a}\,\underbrace{[ire^{i\theta}]}_{z'(\theta)}\,d\theta$$

$$= \text{ave above} \checkmark$$

Lemma: If f is analytic on disc and $|f|$ is constant there, then f is constant.

Pf: $f = u + iv$ $\qquad |f| = \sqrt{u^2+v^2} = c$, a constant

square it: $u^2 + v^2 = c^2 \quad (*)$

C-R eqns?

$$\frac{\partial}{\partial x}(*) : \quad 2u\,u_x + 2v\,v_x = 0$$
$$\frac{\partial}{\partial y}(*) : \quad 2u\,u_y + 2v\,v_y = 0$$

$$\begin{bmatrix} u_x & v_x \\ \underset{\substack{\| \\ -v_x}}{u_y} & \underset{\substack{\| \\ u_x}}{v_y} \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}$$

Aha! $\quad \text{Det} = (u_x)^2 + (v_x)^2 = |\underbrace{u_x + i v_x}_{\text{red box \#1}}|^2 = |f'|^2$

Case $c = 0$. Then $|f| \equiv 0 \Rightarrow f \equiv 0$ ✓

Case $c \neq 0$ $\quad \sqrt{u^2+v^2} = c \Leftarrow \begin{pmatrix} u \\ v \end{pmatrix}$ not the zero vector!

$$\left[A\vec{x} = \vec{0} \text{ with } \vec{x} \neq 0 \Rightarrow \det A = 0. \right]$$

Conclude $f' \equiv 0$. So f is constant. ✓