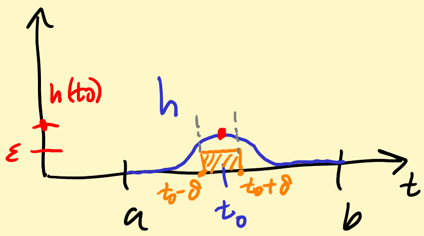
Lecture 17 Averaging property and maximum principle for analytic fcns & harmonic fcns

Easy calculus lemma $h : [a,b] \to \mathbb{R}$ continuous and $h(t) \geq 0$ for all t.

If $\int_a^b h(t)\,dt = 0$, then $h(t) \equiv 0$.

Ex: $\int_0^{2\pi} \sin t \, dt = 0$ does not imply that $\sin t \equiv 0$.

Pf of lemma



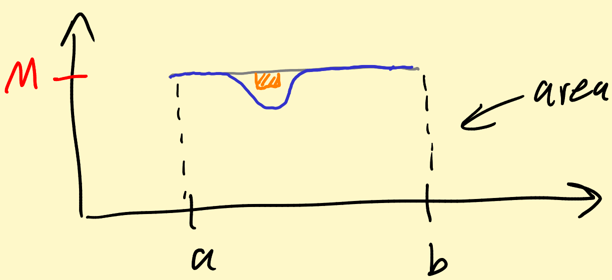
Suppose $h(t_0) > 0$. Let $\varepsilon = \frac{h(t_0)}{2}$. There is a $\delta > 0$ such that $|h(t) - h(t_0)| < \varepsilon$ when $|t - t_0| < \delta$.

See box of area $\varepsilon \cdot (2\delta)$ under curve. So $\int_a^b h(t)\,dt > \varepsilon(2\delta) > 0$ $\lightning$.

Upside down version $h : [a,b] \to \mathbb{R}$ and $0 \leq h(t) \leq M$.

If $\int_a^b h(t)\,dt = M(b-a)$, then $h(t) \equiv M$.

Pf: Replace h by $M - h$ in first version.



First case $D_r(a)$. Suppose f analytic on $D_R(a)$ with $R > r$. Let $M = \text{Max}\{|f(z)| : |z-a| \leq r\}$. If

$|f(a)| = M$, then f is constant on $D_r(a)$.

Pf: $\left|f(a)\right| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(a+re^{it})\,dt\right|$

$$M = |f(a)| \le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(a+re^{it})|}_{0\le h(t)\le M}\,dt \le \frac{1}{2\pi}\cdot M\cdot(2\pi)$$

See $\int_0^{2\pi} h(t)\,dt = M(2\pi - 0)$. So $h(t) \equiv M$.

so $|f(a+re^{it})| \equiv M$, $0 \le t \le 2\pi$.

Aha! Same holds for any radius ρ with $0<\rho<r$.

Conclude that $|f| \equiv M$ on $D_r(a)$.

Lemma $\Rightarrow$ $f \equiv$ const.

Max modulus thm Suppose Ω is a bounded domain and f is a continuous complex valued fcn on the closure $\overline{\Omega}$ of Ω which is analytic on Ω. Then $M = \text{Max}\{|f(z)|;\ z \in \overline{\Omega}\}$ is assumed at some pt z_0 in bndry.

closed and bounded,
$|f|$ cont on $\overline{\Omega}$,
so max is assumed in $\overline{\Omega}$.

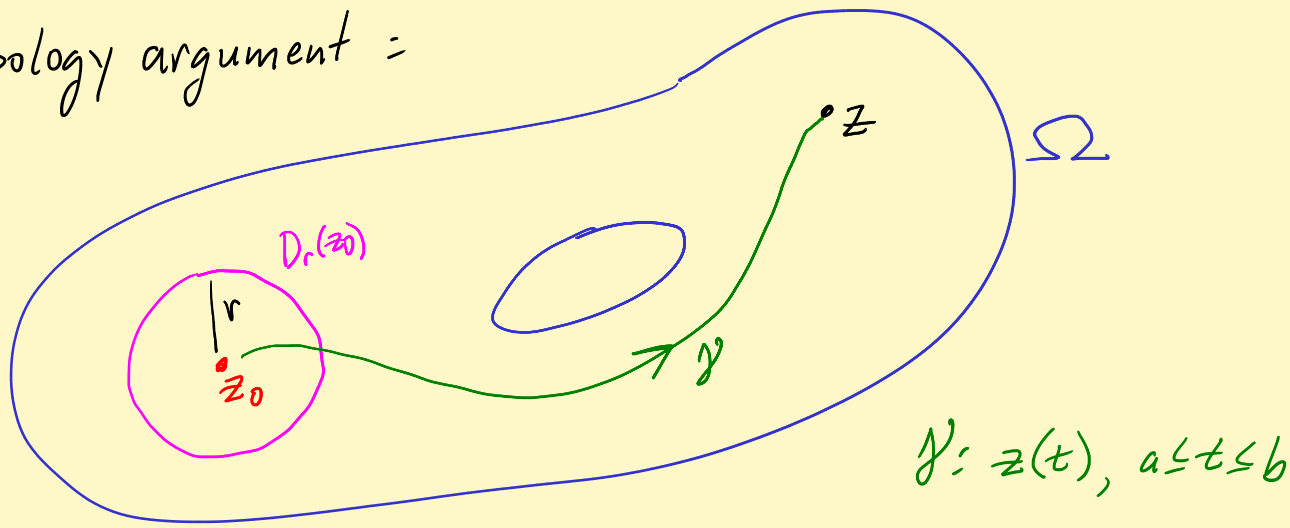
Pf Say max is assumed at $z_0 \in \overline{\Omega}$. $|f(z_0)| = M$.

Case $z_0 \in b\Omega = \partial\Omega \leftarrow$ the boundary of Ω. Done!

Case $z_0 \in \Omega$. Ω open $\Rightarrow \overline{D_r(a)} \subset \Omega$, $r > 0$.

Disc case above shows $|f| \equiv M$ on $D_r(a)$.

Topology argument:

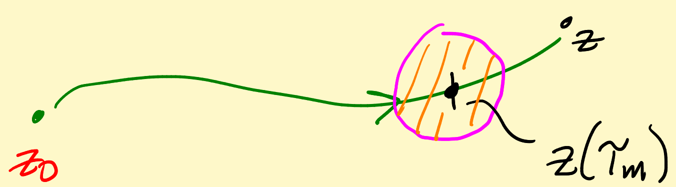


Note $|f(z(t))| \equiv M$ for $t \in [a, a+\delta)$ for some $\delta > 0$.

Let $\tilde{\tau}_m = \max\{\tilde{\tau} : |f(z(t))| \equiv M$ for $a \le t \le \tilde{\tau}\}$.

Claim $\tilde{\tau}_m = b$.

Why. We know $\tilde{\tau}_m > a$. If $\tilde{\tau}_m < b$.



$|f(z(\tilde{\tau}_m))| = M$

$|f| \equiv M$ on disc about $z(\tilde{\tau}_m)$.

$\tilde{\tau}_m$ is not the max. ↯

Conclude $|f| \equiv M$ on Ω. By continuity, $|f| \equiv M$ on

$\overline{\Omega}$ too, so max max is attained on bndry. ✓

Fancier version If $|f|$ assumes a local max at $z_0 \in \Omega$ (Ω doesn't have to be bounded), then f is constant on Ω.

Pf If z_0 is a local max, get disc $\overline{D_r(z_0)} \subset \Omega$ where $|f(z_0)|$ is max on $D_r(z_0)$. Then, from above work, we know $f \equiv$ const on $D_r(z_0)$.

Big fact (coming soon) $f \equiv 0$ on $D_r(z_0) \subset$ domain Ω, then $f \equiv 0$ on Ω.

Apply big fact to $f - c$ if $f \equiv c$ on $D_r(z_0) \subset \Omega$.

Is there a minimum modulus princ? No!

$f(z) = z^2$ has a min modulus at $z = 0$, but is not constant on $\mathbb{C}$.

But, if f is a non-vanishing analytic fcn on a domain Ω, there is a min modulus thm.

Pf: Apply max mod princ to $1/f$.

We've seen : ave prop $\Rightarrow$ max principle.

Claim Harmonic fcns satisfy the ave prop.

Easy Suppose u harmonic on $D_R(a)$. Get harmonic conjugate v for u on $D_R(a)$. So $f = u + iv$ analytic. If $0 < r < R$, we know

$$f(a) = u(a) + i\,v(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})\,dt$$

$$= \underbrace{\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})\,dt}\, + i\,\frac{1}{2\pi}\int_0^{2\pi} v(a + re^{it})\,dt$$

equate real parts. ave prop for u ✓

Straightforward to repeat arguments for analytic fcns to get a max princ for harmonic fcns.

Get a min princ too by applying max princ to $-u$.

Nice trick $f = u + iv$

$$|e^f| = |e^u e^{iv}| = \underbrace{|e^u|}_{e^u}\ \underbrace{|e^{iv}|}_{=1} = e^u$$

u has a local max at $z_0 \Rightarrow |e^f|$ has local max at z_0

$\Rightarrow$ e^f const. Does that imply $u \equiv$ const?

Hmmm. $\underbrace{|e^f|}_{\text{const } c \neq 0} = e^u$. Yes! $u \equiv \operatorname{Ln} c$. ✓