# Lecture 18 More about harmonic fcns

Useful fact: If $u$ is $C^2$-smooth real valued harmonic fcn on a domain $\Omega$, then $f = \underbrace{u_x}_{U} - i u_y$ is analytic on $\Omega$.

$V = -u_y$

Why $u_x$ and $u_y$ are $C^1$-smooth.

C-R eqns?
$$\begin{cases} \dfrac{\partial U}{\partial x} = u_{xx} \overset{?}{=} -u_{yy} = \dfrac{\partial V}{\partial y} \checkmark & \Delta u \equiv 0 \\[2ex] \dfrac{\partial U}{\partial y} = u_{yx} \overset{?}{=} u_{xy} = -\dfrac{\partial V}{\partial x} \checkmark & \text{because } u \text{ is } C^2 \end{cases}$$

Great consequence If $u$ is a $C^2$-smooth real valued harmonic on a simply connected domain $\Omega$, then $u$ has a harmonic conjugate on $\Omega$.

[Hmmm. If $f = u + iv$ analytic, then

$$f' = \begin{cases} u_x + i v_x & \#1 \\ v_y - i u_y & \#2 \end{cases}$$

$$= \underbrace{u_x - i u_y}_{\text{Aha! Know this is analytic}}$$
]

Big idea Let $F = u_x - i u_y$. Since $\Omega$ is simply connected, we know there is an analytic fcn $f$ on $\Omega$ with $f' = F$.

(In fact, $f(z) = \displaystyle\int_{\gamma_a^z} \underbrace{F(w)}_{u_x - i u_y}\, dw$. )

Write $f = U + iV$.

$$f' = \begin{cases} \tilde{U}_x + i\tilde{V}_x \\ \tilde{V}_y - i\tilde{U}_y \end{cases} = u_x - i u_y$$

Aha! $\nabla \tilde{U} = \nabla u$ on $\Omega$. So they differ by a const $c$.

$u = \tilde{U} + c$.

Claim $\tilde{V}$ is a harmonic conjugate for $u$.

$$u + i\tilde{V} = (\tilde{U} + c) + i\tilde{V} = \underbrace{(\tilde{U} + i\tilde{V})}_{f} + c \quad \text{analytic} \checkmark$$

Liouville's theorem for harmonic fcns Suppose $u$ is harmonic on $\mathbb{C}$ and bounded there. Then $u$ must be constant.

Pf: Get harmonic conjugate $v$ for $u$ on $\mathbb{C}$.

$f = u + iv$ is entire. Trick $e^f = e^{u+iv} = e^u e^{iv}$

$u \le M$

$$\underset{\substack{\uparrow \\ \text{entire fcn}}}{|e^f|} = \underbrace{|e^u|}_{e^u} \underbrace{|e^{iv}|}_{=1} = e^u \le e^M$$

$e^f$ bndd entire. Liouville's $\Rightarrow$ $e^f \equiv c$ ← complex const.

Claim $u$ is constant too. $e^u = |e^f| \equiv |c|$

So $u \equiv \operatorname{Ln}|c|$. ✓

Remark Pf shows: If $u$ is bndd above, then $u \equiv c$.

Apply to $-u$ in place of $u$. See $u$ bndd below

$\Longrightarrow$ $u$ const too.

Question Is there a "Cauchy integral formula" for harmonic fcns? Yes! Poisson integral formula.

(Short expository paper on home page: "Something about Poisson and Dirichet")

Suppose $u$ is harmonic on $D_R(0)$ with $R>1$.

Get analytic $f = u + iv$ on $D_R(0)$. Let $a \in D_1(0)$.

$C_1(0)$: $z(t) = 0 + 1 \cdot e^{it}$, $0 \le t \le 2\pi$

Know $$f(a) = \frac{1}{2\pi i} \int_{C_1(0)} \frac{1}{z-a} f(z)\, dz$$

Want to take real part to get formula for $u(a)$, but it's a mess involving both $u$ and $v$.

**Trick**

$$\frac{1}{2\pi i}\int_{C_1(0)} \underbrace{\frac{\bar{a}}{1-\bar{a}z} f(z)}\, dz = 0$$

analytic inside and on $C_1(0)$.

$1-\bar{a}z = 0$

$z = 1/\bar{a}$

$|a|<1$

$|\bar{a}|=|a|$

$|1/\bar{a}| > 1$

Cauchy thm!

**Miracle:**

$$f(a) = \frac{1}{2\pi i}\int_{C_1(0)} \Big[\frac{1}{z-a} - \underbrace{\frac{\bar{a}}{1-\bar{a}z}\Big] f(z)\, dz}_{\text{subtacting } 0}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \left(\frac{1}{e^{it}-a} - \frac{\bar{a}}{1-\bar{a}e^{it}}\right) f(e^{it})\, ie^{it}\, dt$$

$$e^{it} = \frac{1}{e^{-it}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{e^{it}}{\underbrace{e^{it}-a}} - \frac{\bar{a}}{\underbrace{e^{-it}-\bar{a}}}\right) f(e^{it})\, dt$$

conjugates!

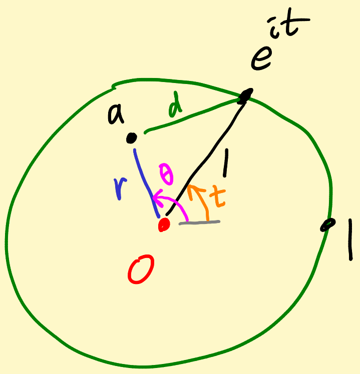
$$= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{1-|a|^2}{|e^{it}-a|^2}}_{\text{real!}} f(e^{it})\, dt$$

Can take real part!

$$\boxed{u(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|a|^2}{|e^{it}-a|^2}\, u(e^{it})\,dt}$$

Poisson formula

$a = re^{i\theta}$




Law of cosines

$d^2 = 1^2 + r^2 - 2r\cdot 1\cos(\theta - t)$

$$\boxed{u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\, u(e^{it})\,dt}$$

Do it on a circle of radius $R$ about $z_0$ =

$$u(z_0+re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2+r^2-2rR\cos(\theta-t)}\, u(z_0+Re^{it})\,dt$$

Dirichlet problem  Given a continuous real valued fcn $\varphi$ on $[0,2\pi]$ with $\varphi(0)=\varphi(2\pi)$, find a fcn $u$ that is cont on $\overline{D_1(0)}$, harmonic in $D_1(0)$, with $u(e^{i\theta}) = \varphi(\theta)$.

<u>Exciting fact</u> Poisson formula <u>solves</u> the D. prob!

$$u(re^{i\theta}) = \int_0^{2\pi} \underbrace{P(r,\theta,t)}_{} \varphi(t)\, dt$$

Poisson kernel $= \dfrac{1}{2\pi} \dfrac{1-r^2}{|e^{it} - re^{i\theta}|^2}$