Lecture 20 Power series   Review on Monday

Thm A power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has a radius of convergence R

$0 \le R \le \infty$ with properties

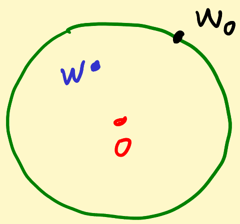
1) series converges absolutely in $D_R(z_0)$
2) converges uniformly on $D_r(z_0)$ when $0<r<R$
3) diverges when $|z-z_0|>R$
4) converges to an analytic fcn $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ on $D_R(z_0)$.

Pf: Step 1 If series converges at $w_0 \neq z_0$, then $\sum_{n=0}^{\infty} a_n(w_0-z_0)^n$ converges. [Take $z_0=0$.]

Then $a_n w_0^n \longrightarrow 0$ as $n \to \infty$. Let $\varepsilon = 1$.

There is an $N_0 \in \mathbb{N}$ such that $|a_n w_0^n| < 1$ when $n > N_0$. (So $|a_n w_0^n|$ is a bounded sequence.)

Step 2 Compare tail end to a convergent geom series.



Claim Series converges inside disc of radius $|w_0|$.

$$|a_n w_0^n| < 1 \quad \text{for} \quad n > N_0.$$

$$|a_n w^n| = |a_n w^n| \frac{|w_0^n|}{|w_0|^n} = \underbrace{|a_n w_0^n|}_{<1} \underbrace{\left|\frac{w}{w_0}\right|}_{<1}^n$$

$$< \left|\frac{w}{w_0}\right|^n \quad \text{when } n > N_0.$$

Aha! $\sum_0^\infty a_n w^n$ is absolutely convergent by comparing tail end to $\sum_{n>N_0} \left|\frac{w}{w_0}\right|^n$

Note All I needed to do this was that the tail end of series at w_0 was bounded.

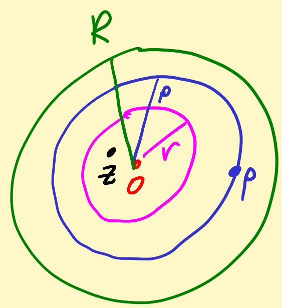
$$R = \sup\{r \geq 0 : \{a_n r^n\}_{n=0}^\infty \text{ is a bounded seq}\}$$

This is the radius on conv.

Step 3 Series converges uniformly in $D_r(0)$ when $0 < r < \rho < R$.

$z_0 = 0$

$$f(z) = \sum_{n=0}^{N} a_n z^n + \underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{\mathcal{E}_N(z)}$$



$$|a_n z^n| \cdot \frac{\rho^n}{\rho^n} = \underbrace{|a_n \rho^n|}_{<M} \left|\frac{z}{\rho}\right|^n < M \left|\frac{z}{\rho}\right|^n < M\left(\frac{r}{\rho}\right)^n$$

$$r|z| = |rz|$$
$$|z||w| = |zw|$$

$$|\varepsilon_N(z)| \leq \sum_{n=N+1}^{\infty} M\left(\frac{r}{\rho}\right)^n$$

$$= M\left(\frac{r}{\rho}\right)^{N+1} \underbrace{\left[1 + \left(\frac{r}{\rho}\right) + \left(\frac{r}{\rho}\right)^2 + \cdots\right]}_{\frac{1}{1-\frac{r}{\rho}}}$$

$$|\varepsilon_N(z)| \leq \frac{M\left(\frac{r}{\rho}\right)^{N+1}}{1-\frac{r}{\rho}} \longrightarrow 0$$ uniformly in z as $N \to \infty$.

Property 3 : Def^n of RoC in terms of boundedness of terms in series $\Rightarrow$ series can't converge outside R.

Property 4 Use Morera's on $D_r(z_0)$ with $0 < r < R$.

Important fact from MA 341 A series of continuous fcns that converges uniformly convergences to a continuous fcn.

So $f(z) = \sum_0^\infty a_n (z-z_0)^n$ is continuous in $D_r(z_0)$.

Let γ be a closed curve in $D_r(z_0)$.

$$S_N(z) = \sum_0^N a_n (z-z_0)^n$$

$$\left|\int_\gamma f\,dz\right| = \left|\int_\gamma f\,dz - \int_\gamma S_N\,dz + \underbrace{\int_\gamma S_N\,dz}_{=0}\right|$$

$=0$ by Fund thm calc. S_N is a poly!

$$= \left|\int_\gamma (f - S_N)\,dz\right|$$

$$\le \underbrace{\left(\max_\gamma |f - S_N|\right)}_{\to 0 \text{ as } N\to\infty} \text{Length}(\gamma)$$

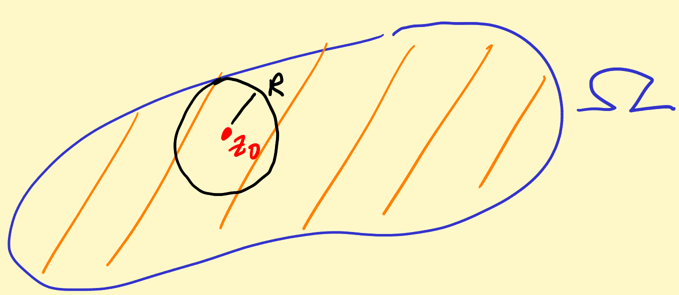
$\to 0$ as $N \to \infty$ by unif conv on $D_r(z_0)$.

Conclude $\int_\gamma f\,dz = 0$. Morera's $\Rightarrow$ f analytic!

Remark Today showed power series $\Rightarrow$ analytic fcns.

After exam, will show analytic fcns = power series given by Taylors formula.

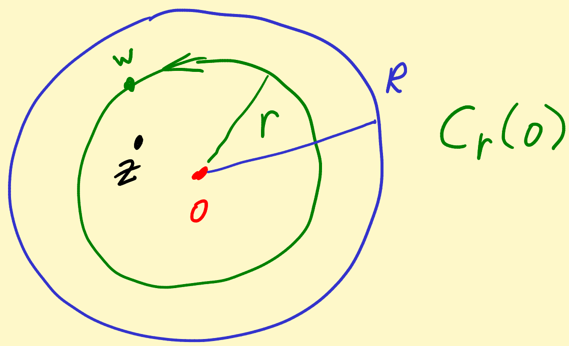
Plan



f analytic on Ω

Will get a power series for f about z_0 with RoC $\geq R$. Let r be such that $0 < r < R$.

Let $z_0 = 0$. Cauchy integral formula using $C_r(0)$:



Aha! $\left|\frac{z}{w}\right| < 1$

$$f(z) = \frac{1}{2\pi i}\int_{C_r(0)} \underbrace{\frac{f(w)}{w-z}}_{\frac{f(w)}{w}\cdot\frac{1}{1-\frac{z}{w}}}\,dw$$

$$\frac{1}{1-\left(\frac{z}{w}\right)} = 1 + \left(\frac{z}{w}\right) + \left(\frac{z}{w}\right)^2 + \cdots + \left(\frac{z}{w}\right)^N + \underbrace{\frac{\left(\frac{z}{w}\right)^{N+1}}{1-\frac{z}{w}}}_{\mathcal{E}_N\left(\frac{z}{w}\right)}$$

$$f(z) = \sum_{n=0}^{N} \underbrace{\left(\frac{1}{2\pi i}\int_{C_r(0)} \underbrace{\frac{f(w)}{w}\cdot\frac{1}{w^n}}_{\frac{f(w)}{w^{n+1}}}\,dw\right)}_{\frac{f^{(n)}(0)}{n!}} z^n + E_N(z)$$

$\frac{f^{(n)}(0)}{n!} \leftarrow$ Taylor coeff! From Higher order Cauchy int formula.

Easy to estimate error

$$\left| E_N(z) \right| = \left| \frac{1}{2\pi i} \int_{C_r(0)} \frac{f(w)}{w} \frac{\left(\frac{z}{w}\right)^{N+1}}{1-\frac{z}{w}} \, dw \right|$$

Basic estimate. $\left|\frac{z}{w}\right| = \frac{|z|}{r} \leftarrow$ Hmmm. Need $|z| < \rho < r$ so that $\left|\frac{z}{w}\right| < \frac{\rho}{r} < 1$.

$|f(z)| < M$ on $C_r(0)$.

Get $\left| E_N(z) \right| < \frac{1}{2\pi} \frac{M}{r} \cdot \frac{\left(\frac{\rho}{r}\right)^{N+1}}{1-\frac{\rho}{r}} \cdot (2\pi r)$

$\longrightarrow 0$ uniformly on $D_\rho(0)$.