Lecture 22 Zeroes of analytic functions

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Suppose z_0 is a zero of $f(z)$. So $f(z_0) = a_0 = 0$.

Let a_N be the first non-zero coeff.

[Then $f^{(n)}(z_0) = 0$ for $n = 1, 2, \ldots, N-1$.]

Note: If all $a_n = 0$, then $f \equiv 0$ inside RoC of power series.

$$f(z) = a_N(z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$$

$$= (z-z_0)^N \left[\underbrace{a_N + a_{N+1}(z-z_0) + a_{N+2}(z-z_0)^2 + \cdots}_{F(z)} \right]$$

Hmmm.

Important!

[$F(z)$ has a power series expansion and this expansion converges at exactly same z as the series for $f(z)$.

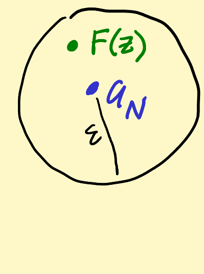
Consequently the radii of convergence for f and F series are the same!]

Notice that $F(z_0) = a_N + 0 + 0 + \cdots = a_N \neq 0$.

Aha! Zero of $f(z)$ at z_0 is an isolated zero, meaning that there is a $\delta > 0$ such z_0 is the only zero of $f(z)$ in $D_\delta(z_0)$.

Why F = power series $\Rightarrow$ F analytic $\Rightarrow$ F continuous.

Let $\varepsilon = \dfrac{|a_N|}{2}$.



Take $\delta > 0$ such that $|F(z) - a_N| < \varepsilon$ when $|z - z_0| < \delta$

so that $|F(z)| \neq 0$ in $D_\delta(z_0)$.

$$f(z) = (z - z_0)^N \underbrace{F(z)}_{\neq 0 \text{ on } D_\delta(z_0)}$$

zero only at z_0.

Def^n f has zero of multiplicity N (or order N) at z_0.

Lemma If f is analytic on $D_R(z_0)$, then either

A) $f \equiv 0$ on $D_R(z_0)$, or

B) f has an isolated zero at z_0 of finite multiplicity N.

Pf Done!

<u>Identity theorem</u> Suppose f is analytic on a <u>domain</u> Ω.

Let Z_f = zero set of f = $\{z \in \Omega : f(z) = 0\}$.

If Z_f has a <u>limit point</u> $w_0 \in \Omega$, then $f \equiv 0$.

[w_0 is a limit pt of Z_f means there exist a sequence of pts $z_n \in Z_f$ with $z_n \neq w_0$ for all n and $\lim_{n\to\infty} z_n = w_0$. Think: Zeroes "pile up" at w_0.]

Identity thm way false $\mathbb{R} \to \mathbb{R}$:

$$h(t) = \begin{cases} 0 & t = 0 \\ e^{-1/t^2} \sin\frac{1}{t} & t \neq 0 \end{cases} \qquad \mathcal{C}^\infty\text{-smooth on } \mathbb{R}.$$

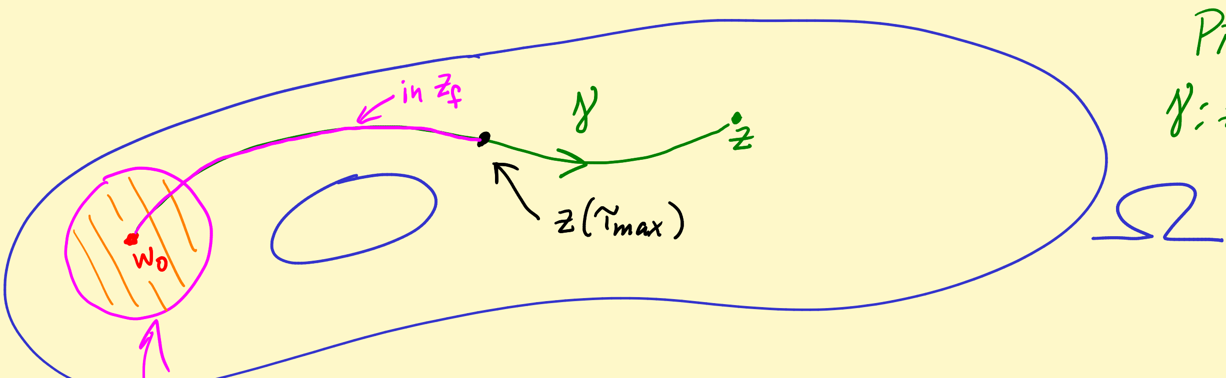
zeroes pile up at $t=0$, but $h \not\equiv 0$.

<u>Pf</u> of identity thm.

Suppose w_0 is a lim pt of Z_f in Ω.

Pick any $z \in \Omega$.

$\gamma: z(t),\ a \le t \le b$



Ω open. So $\exists\ D_r(w_0) \subset \Omega$

Lemma $\Rightarrow f \equiv 0$ on $D_r(w_0)$

Let $\tilde{\imath}_{max} = \sup\{\tilde{\imath} : f(z(t)) = 0 \text{ for } a \le t \le \tilde{\imath}\}$

$z(\tilde{\imath})$ in zero set of f by continuity

Note: $\tilde{\imath}_{max} > a$ since $f \equiv 0$ on $D_\delta(w_0)$.

Claim: Lemma will show $\tilde{\imath}_{max} = b$. So $f(z) = 0$ too.

z arbitrary, so $f \equiv 0$.

See zeroes of f pile up along γ at $z(\tilde{\imath}_{max})$. Lemma $\Rightarrow$

$\exists\ D_\delta(z(\tilde{\imath}_{max}))$ where $f \equiv 0$. Hmmm.

$\tilde{\imath}_{max}$ is not the max. $\lightning$ unless $\tilde{\imath}_{max} = b$.

Fun consequences Only one way to extend e^x (or $\sin x$ or $\cos x$ etc) from $\mathbb{R}$ to $\mathbb{C}$ as an analytic fcn.

Why: If $E_1(z)$ and $E_2(z)$ both extend e^x, then

$E_1 - E_2$ is an entire fcn that vanishes on $\mathbb{R}$.

Every pt in $\mathbb{R}$ is a limit pt of $\mathbb{R}$ in $\Omega = \mathbb{C}$.

Ident thm $\Rightarrow$ $E_1 - E_2 \equiv 0$ on $\mathbb{C}$. ✓

Fact Any trig identity holds for complex angles.

$$\sin(\alpha + \beta) = \cos\alpha \sin\beta + \sin\alpha \cos\beta$$

Step 1. Let β go complex. $\beta = z$. See true for $\beta = z \in \mathbb{C}$.

Step 2. Fix $\beta \in \mathbb{C}$. Now α go complex. Done.

<u>Big fact</u> Can differentiate convergent power series term by term:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + \cdots$$

$$f'(z) = 0 + a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots$$

$$= \sum_{n=1}^{\infty} n a_n (z-z_0)^{\underbrace{n-1}_{m}}$$

$$= \sum_{m=0}^{\infty} (m+1)\, a_{m+1} (z-z_0)^m = \sum_{m=0}^{\infty} b_m (z-z_0)^m$$

$$b_m = \frac{(f')^{(m)}(z_0)}{m!} = \frac{(m+1)}{(m+1)} \frac{f^{(m+1)}(z_0)}{m!} = (m+1)\underbrace{\frac{f^{(m+1)}(z_0)}{(m+1)!}}_{a_{m+1}}$$

where the RoC of series for f' is the <u>same</u> as the RoC of series for f.

<u>Why</u>: $f'(z)$ is analytic where f is. so $R_{f'} \geq R_f$.

Hmmm. How could $R_{f'} > R_f$? If it were, the series would converge to $\tilde{g}$ analytic on bigger disc and $\tilde{g} = f'$ on D_{R_f}. Let F be an antiderivative of $\tilde{g}$ on bigger disc. Note $f = F + C$ on D_{R_f} because F and f are both antiderivatives of $g = f'$ there.

Power series of f has same RofC as series for $F+C$, which is bigger than R_f. ↯.