# Lecture 23 Differentiating power series

Last time: $f(z) = a_0 + \sum_{n=1}^{\infty} a_n (z-z_0)^n$ $\quad RoC = R_f$

$$f'(z) = 0 + \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} \qquad RoC = R_{f'}$$

$$\sum_{n=0}^{\infty} b_n (z-z_0)^n = \underline{\sum_{\eta=0}^{\infty} (\eta+1) a_{\eta+1} (z-z_0)^{\eta}} \qquad \eta = n-1$$

Know $f'$ is analytic where $f$ is. So $R_{f'} \geq R_f$.

Last time: Showed if $R_{f'} > R_f$.

$\downarrow$

converges to $\tilde{F}$ $\leftarrow$ take antiderivative $F$

See $F = f + C$ on $R_f$ disc.

Get bigger RoC for $f$. $\lightning$

Conclude $\boxed{R_{f'} = R_f.}$

Final step: $b_n = \dfrac{(f')^{(n)}(z_0)}{n!} = \dfrac{(n+1)}{(n+1)} \dfrac{f^{(n+1)}(z_0)}{n!}$

$$= (n+1) \underbrace{\frac{f^{(n+1)}(z_0)}{(n+1)!}}_{a_{n+1}} = (n+1) a_{n+1} \checkmark$$

(Replace $n$ by $\eta$, see formula above.)

EX Fun example. Fibonnacci #'s $\quad a_0 = 1,\ a_1 = 1,\ a_n = a_{n-1} + a_{n-2}$, $n \geq 2$.

$1, 1, 2, 3, 5, 8, \ldots$

Power series: $\quad f(z) = \sum_{n=0}^{\infty} a_n z^n \longleftarrow$ "Generating function"

Claim: RoC is $\frac{\sqrt{5}-1}{2}$ (which is related to the "Golden ratio")

Hmmm. Assume for a moment that the series has RoC $R>0$.

$$
\begin{aligned}
f(z) &= a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \\
z f(z) &= \qquad a_0 z + a_1 z^2 + \cdots + a_{n-1} z^n + a_n z^{n+1} + \cdots \\
z^2 f(z) &= \qquad\qquad a_0 z^2 + \cdots + a_{n-2} z^n + a_{n-1} z^{n+1} + \cdots
\end{aligned}
$$

Aha!

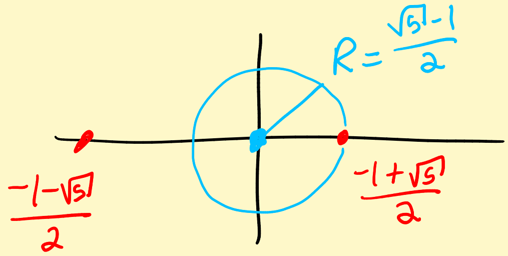
$$f(z) - z f(z) - z^2 f(z) = \underset{1}{a_0} + (\underset{1}{a_1} - \underset{1}{a_0}) z + 0 + 0 + 0 + \cdots$$

If $R>0$, $\quad f(z) = \frac{1}{1-z-z^2}$ on a disc about zero.

Aha! Let $F(z) = \frac{1}{\underbrace{1-z-z^2}}$.

zeroes $\quad \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)(1)}}{2(-1)} = \frac{1 \pm \sqrt{5}}{-2}$

$\frac{-1 \pm \sqrt{5}}{2}$




RoC of power series for $F$ is $\frac{\sqrt{5}-1}{2}$.

Aha! $\underbrace{F(z)}\,(1-z-z^2) = 1$

$$\sum_{n=0}^{\infty} b_n z^n \,(1-z-z^2) = 1$$

See the $b_n$'s satisfy the Fibonnacci recursion formula!

They are = to the $a_n$'s.

Key fact: Power series coeff are unique. They are given by Taylor's formula.

Two analytic fcns: $f(z) = \sum_0^\infty a_n (z-z_0)^n$

$$g(z) = \sum_0^\infty b_n (z-z_0)^n$$

If $f \equiv g$ on $D_\varepsilon(z_0)$, then all $a_n = b_n$'s.