Lecture 24 Hadamard's formula

Radius of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ exists and equals R where

$$\frac{1}{R} = \operatorname{Lim\,Sup}_{n\to\infty} \sqrt[n]{|a_n|}$$

Def^n of LimSup

$$= \lim_{N\to\infty} \underbrace{\operatorname{Sup}\{ \underbrace{\sqrt[n]{|a_n|}}_{\geq 0} : n \geq N \}}_{} \geq 0$$

can only decrease or stay same as we toss away a_n's.

non-increasing seq.

Sup = "Least upper bound"

$R=0$ | $R=\infty$

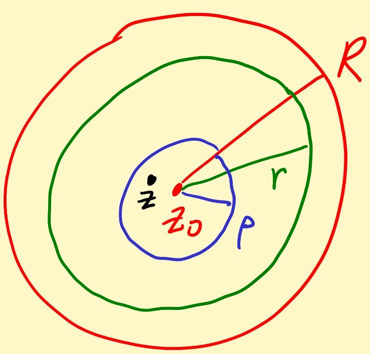
$LS=+\infty$ | $LS=0$

allowed

Fact MA 341: A non-increasing seq bounded below has a limit.

[Special case LimSup $=\infty$. Seq is $\infty, \infty, \infty, \ldots$]

Think Those Sup's are sliding <u>down</u> to $\frac{1}{R}$.



$$0 < \rho < r < R$$

$$\frac{1}{R} < \frac{1}{r} < \frac{1}{\rho}$$

$\operatorname{Sup}\{\sqrt[n]{|a_n|} : n \geq N\}$ sliding down to $\frac{1}{R}$

So, $\exists\, N_0 \in \mathbb{N}$ such that

$$\operatorname{Sup}\{\sqrt[n]{|a_n|} : n \geq N_0\} < \frac{1}{r}$$

$\frac{1}{r}$ is an upper bound

So $\sqrt[n]{|a_n|} < \frac{1}{r}$ for $n \geq N_0$.

n-th power: $|a_n| < \frac{1}{r^n}$ for $n \geq N_0$

Multiply by $|z|^n$: $|a_n z^n| < \frac{|z|^n}{r^n} < \left(\frac{\rho}{r}\right)^n$

take $z_0 = 0$

< 1 ✓

Aha! Power series converges by comparison with geom series!

[Can also deduce that the power series <u>converges</u> uniformly on $D_\rho(z_0)$.]

Last step: Show $\sum_0^\infty a_n z^n$ diverges when $\underbrace{|z| > R}_{\frac{1}{|z|} < \frac{1}{R}}$.

$$\frac{1}{|z|} < \frac{1}{R} \leq \sup\{\sqrt[n]{|a_n|} : n \geq N\}$$

!

So, for <u>each</u> N, there exists an $n \in \mathbb{N}$ such that

$$\frac{1}{|z|} < \sqrt[n]{|a_n|}$$

(otherwise $\frac{1}{|z|}$ is an upper bound that is <u>less</u> than the least up. bnd.!)

n-th power: $\frac{1}{|z|^n} < |a_n|$

Aha! $1 < |a_n z^n|$ for this $n > N$.

terms of sum $\sum_0^\infty a_n z^n$ don't $\longrightarrow 0$.

Series diverges. ✓

Rule of thumb: Check Ratio test, root test, Hadamard in that order.

EX $\quad a_0 + a_3 z^3 + a_6 z^6 + \cdots$ where

$$a_0 = 1, \quad a_{n+3} = \frac{n(n+1)(2n+13)}{n(n+3)(n+7)} a_n \quad \text{for } n = 3, 6, 9, \ldots$$

Ratio test: $\quad \text{Lim} \left| \frac{u_{\text{way out there}}}{u_{\text{right before}}} \right|$

way out there $\to$ out there

$$= \lim_{n\to\infty} \left| \frac{a_{n+3} z^{n+3}}{a_n z^n} \right|$$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+13)}{n(n+3)(n+7)} |z|^3 = 1 \cdot 1 \cdot 2 \, |z|^3$$

$$= 2|z|^3 \quad \begin{matrix} < 1 & \text{conv} \\ > 1 & \text{div} \end{matrix}$$

$$|z| < \frac{1}{2^{1/3}} \quad \text{conv}$$

$$R = \frac{1}{2^{1/3}}$$

$$|z| > \frac{1}{2^{1/3}} \quad \text{div}$$

EX $\displaystyle\sum_{n=1}^{\infty} z^{n!}$ $\quad a_n = \begin{cases} 1 & n = m! \\ 0 & \text{otherwise} \end{cases}$

So $\sup\{\sqrt[n]{|a_n|} : n \geq N\} = 1$ for all N.

$$\frac{1}{R} = \lim_{N\to\infty} \sup = 1. \quad \text{So } R = 1.$$

EX $\displaystyle\sum_{n=0}^{\infty} 2^n z^{n^2}$

Hate formula : If $\displaystyle\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists, then it is equal to the RoC. (True, but series with lots of 0 terms ruin the formula, and this limit often does not exist.)

Use ratio test :

$$\left|\frac{u_{\text{out there}}}{u_{\text{before}}}\right| = \left|\frac{2^{n+1} z^{(n+1)^2}}{2^n z^{n^2}}\right|$$

$$= 2|z|^{(n+1)^2 - n^2} = 2|z|^{2n+1}$$

Hmmm. Want to use ratio test.

Aha! When $|z|>1$. $\lim \to \infty$. Series diverges.

When $|z|<1$. $\lim \to 0 < 1$ Series converges.

Conclude $R=1$.

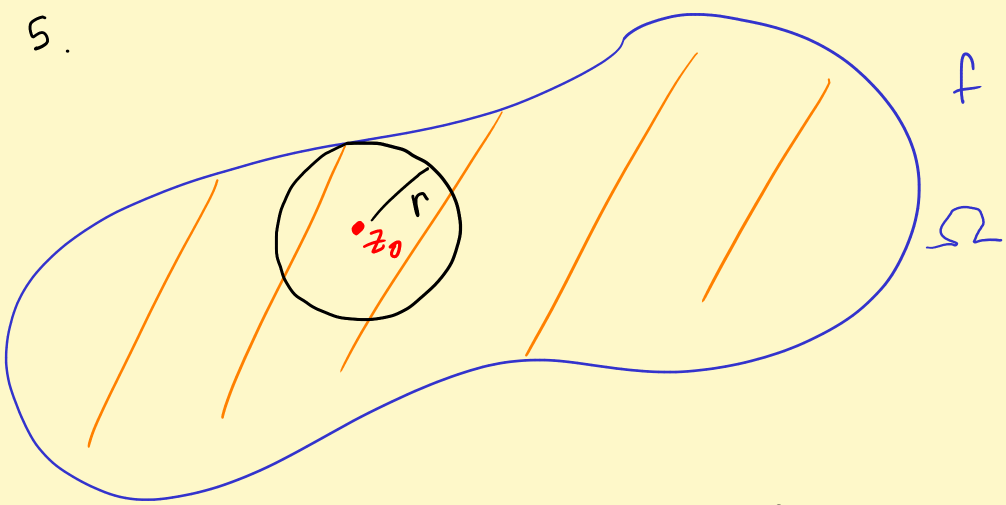
EX $$\sum_{n=0}^{\infty} \Big[\underbrace{2+(-1)^n}_{=1 \text{ or } 3}\Big]^n z^n = 3 + z + 3^2 z^2 + z^3 + 3^4 z^4 + z^5 + \cdots$$

pull apart $$\underbrace{\sum_{n \text{ even}} 3^n z^n + \sum_{n \text{ odd}} z^n}$$

RoC = minimum of 2 radii

$R=\infty$ means $f(z) = \sum a_n (z-z_0)^n$ is entire

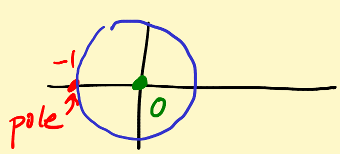
5.2 : 5.



f analytic on Ω

RoC of series for f at z_0 $\geq$ r

EX $\dfrac{1}{1+z}$



$R \geq 1$

pole means $R \leq 1$ too. So $R=1$.