Lecture 25 Uniqueness theorems, Cauchy theorem on an annulus

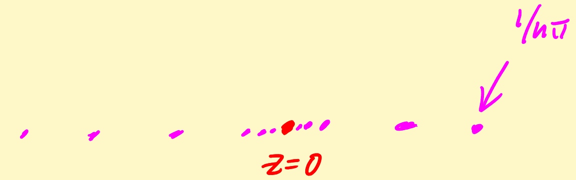
Uniqueness theorem (Identity theorem) f analytic on a domain Ω.

If the zero set $Z_f = \{z \in \Omega : f(z) = 0\}$ has a limit point $z_0 \in \Omega$, then $f \equiv 0$ on Ω.

EX $\sin \frac{1}{z}$ on $\Omega = \mathbb{C} - \{0\}$.

$$\sin w = 0 \iff w = n\pi,\ n \in \mathbb{Z}.$$

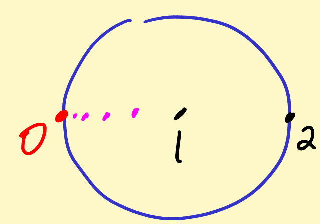
$$\frac{1}{z} = n\pi \qquad z = \frac{1}{n\pi} \qquad (n \neq 0)$$



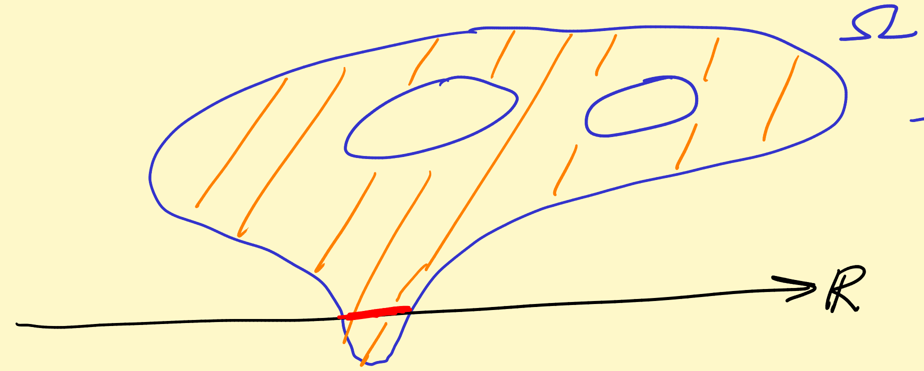
but $\sin \frac{1}{z} \not\equiv 0$.

OK, because 0 is a bndry pt of $\mathbb{C} - \{0\}$.

Or take $\Omega = D_1(1)$



Application



f analytic on Ω and $f(x) = e^x$ on red line.

Zero set of $f(z) - e^z$ has a limit pt in Ω.

So $f(z) \equiv e^z$ on Ω.

Cor: If f is analytic on a domain Ω and $f \equiv 0$ on $D_\varepsilon(z_0) \subset \Omega$, the $f \equiv 0$ on Ω.

Cor: If u is harmonic on a domain Ω and $u \equiv 0$ on $D_\varepsilon(z_0) \subset \Omega$, then $u \equiv 0$ on Ω.

Pf Recall, if u harmonic, then $F(x+iy) = u_x(x,y) - i u_y(x,y)$ is analytic on Ω (C-R equs ✓).

Note that $F \equiv 0$ on $D_\varepsilon(z_0)$. So $F \equiv 0$ on Ω.

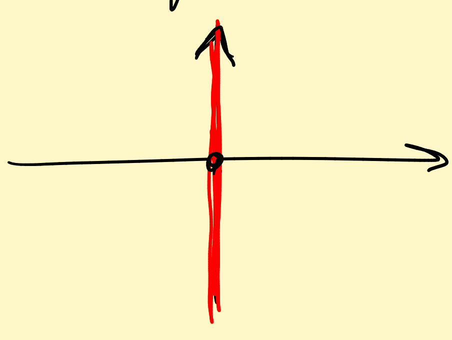
Hmmm. So $u_x \equiv 0$ and $u_y \equiv 0$ on a domain Ω, i.e., $\nabla u \equiv 0$ on Ω. So $u \equiv$ const on Ω.

Since $u(z_0) = 0$, that constant is zero.

Question: Is there Identity theorem for harmonic fcn? (Can the zeroes of harmonic fcns pile up?)

EX: $u(x,y) = \mathrm{Re}[x+iy] = x$ is harmonic on $\Omega = \mathbb{C}$.

Zero set is the imaginary axis.

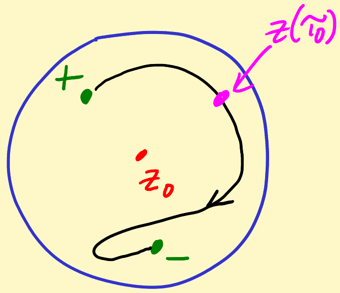


No Identity thm for harmonic fcns.

Question: Can a zero of a harmonic fcn be isolated?

No. (Hint: Ave property)

Suppose z_0 is the <u>only</u> zero of harmonic fcn u in $D_\varepsilon(z_0)$.



<u>Claim</u> u is either always >0 or always <0 in $D_\varepsilon(z_0) - \{z_0\}$.

$\gamma : z(t);\ a \le t \le b$

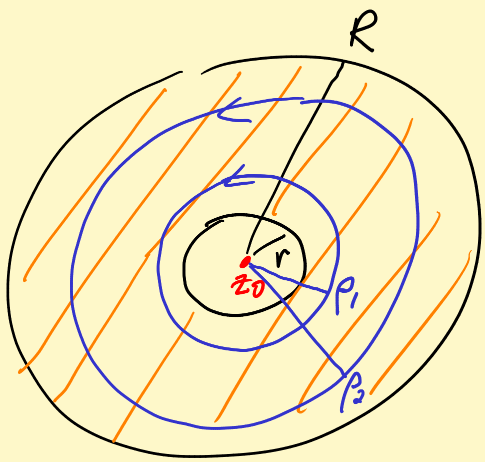
Aha! Apply IVT to $u(z(t))$ on $[a,b]$.

Get $\tilde{\tau}_0 \in (a,b)$ where $u(z(\tilde{\tau}_0)) = 0$. $\lightning$ (z_0 is the <u>only</u> zero in disc.)

$$0 = u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0 + re^{it})}_{\substack{>0 \\ \text{or } <0}}\, dt \quad \text{for } 0 < r < \varepsilon.$$

So $u(z_0) > 0$ or < 0. $\lightning$.

<u>Cauchy theorem for an annulus</u>



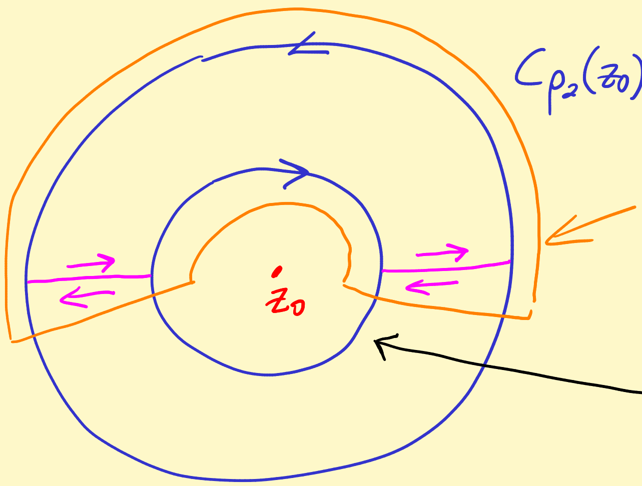
f analytic on
$A(z_0; r, R) = \{z : r < |z - z_0| < R\}$

$r < \rho_1 < \rho_2 < R$

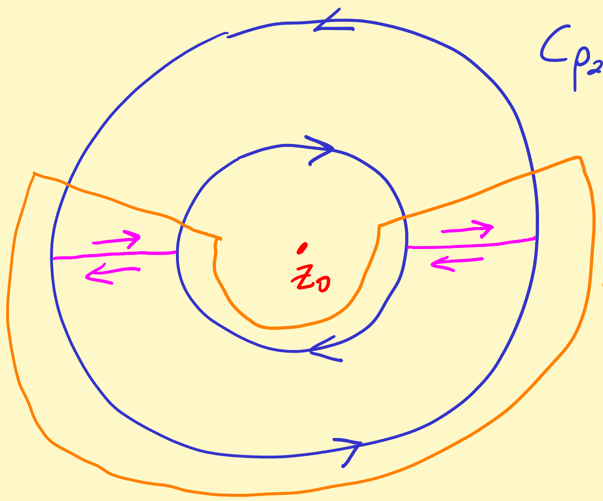
Cauchy theorem: $\displaystyle \left(\int_{C_{\rho_2}(z_0)} - \int_{C_{\rho_1}(z_0)}\right) f\, dz = 0$

counterclockwise circles.

Pf



So $\displaystyle\int_{\gamma_{\text{Top}}} f\,dz = 0.$



Get $\displaystyle\int_{\gamma_{\text{bottom}}} f\,dz = 0$ too.

$$0 = \left(\int_{\gamma_{\text{top}}} + \int_{\gamma_{\text{bottom}}}\right) f\,dz = \left(\int_{C_{\rho_2}(z_0)} + \underbrace{\int_{-C_{\rho_1}(z_0)}}_{-\int_{C_{\rho_1}(z_0)}}\right) f\,dz \quad \checkmark$$

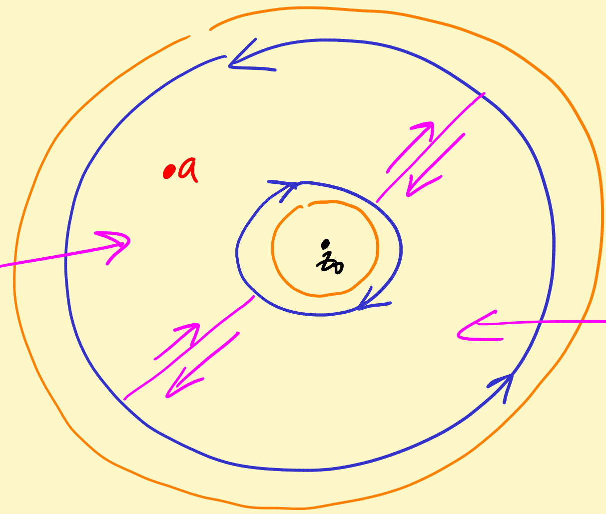
cuts cancel!

Cauchy integral formula for an annulus

Same hyp as before.

$a \in A(z_0; \rho_1, \rho_2)$

Use Cauchy integral formula on this half.



use Cauchy thm on this half since $\frac{f(z)}{z-a}$ is analytic inside and on curve.

$$f(a) = \frac{1}{2\pi i}\left(\int_{C_{\rho_2}(z_0)} - \int_{C_{\rho_1}(z_0)}\right)\frac{f(z)}{z-a}\,dz$$

Isolated singularities of analytic fcns

z_0 is an isolated sing of f if f is analytic on $D_r(z_0) - \{z_0\}$.

EX: a) $\frac{1}{z-z_0}$ blows up at z_0. (So does $\frac{1}{(z-z_0)^3}$)

Has a "pole" at z_0.

b) (Think $\frac{f(z)-f(z_0)}{z-z_0}$ = DQ not defined at z_0.)

$\frac{\sin z}{z}$ ← has a removable sing at $z=0$.