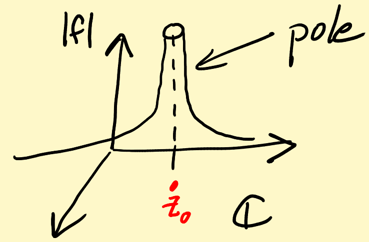
Lecture 26 Laurent expansions

Isolated singularities: f analytic on $D_\varepsilon(z_0) - \{z_0\}$

EX a) $f(z) = \dfrac{1}{(z-z_0)^5}$ has a "pole" at $z = z_0$.

$$\lim_{z \to z_0} f(z) = \infty$$



b) $\dfrac{\sin z}{z} = \dfrac{\sin z - \sin 0}{z - 0}$ analytic on $\mathbb{C} - \{0\}$.

$$= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} = \frac{z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}{z}$$

$$= \underbrace{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots}_{RoC = \infty}$$

converges for exactly same z as series for $\sin z$.

$RoC = \infty$

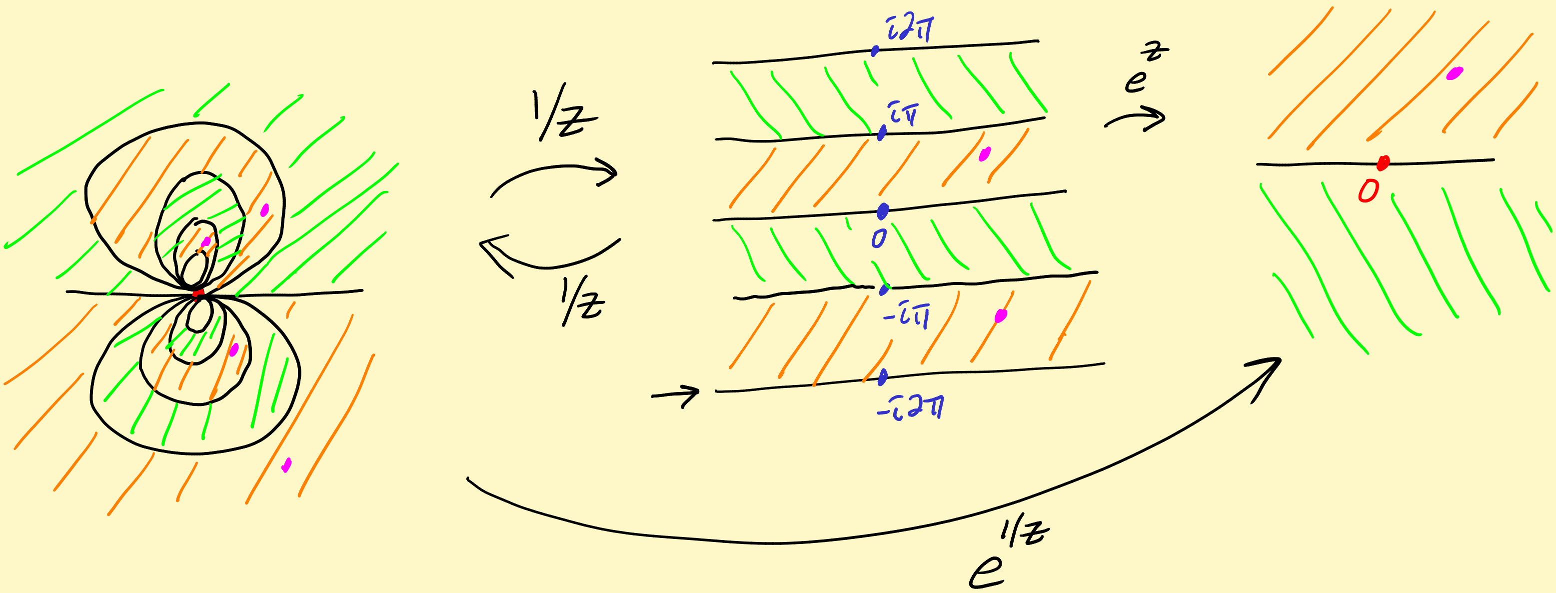
Converges to an entire fcn $H(z)$

$$H(z) = \begin{cases} \dfrac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

Aha! Can make $\frac{\sin z}{z}$ analytic near $z=0$ by giving it the value 1 at $z=0$.

$z=0$ is called a "removable singularity" for $\frac{\sin z}{z}$.

c) $e^{1/z}$ has a horrific singularity at $z=0$!
It shreds the complex plane at $z=0$.



Wow! Now matter how small $\varepsilon > 0$ is, $e^{1/z}$ maps

$D_\varepsilon(0) - \{0\}$ <u>infinite-to-one</u> onto $\mathbb{C} - \{0\}$.

$e^{1/z}$ has an "essential singularity" at $z=0$.

d) $$\frac{\cos z}{z^3} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z^3}$$

residue term

$$= \underbrace{\frac{1}{z^3} - \frac{(1/2!)}{z}}_{\text{rational fcn}} + \underbrace{\frac{1}{4!} z - \frac{1}{6!} z^3 + \cdots}_{\text{regular power series}}$$

called the "principal part" at pole $z=0$.

Get Laurent expansions:

$$\frac{\sin z}{z} = \text{regular power series} \quad \text{(removable)}$$

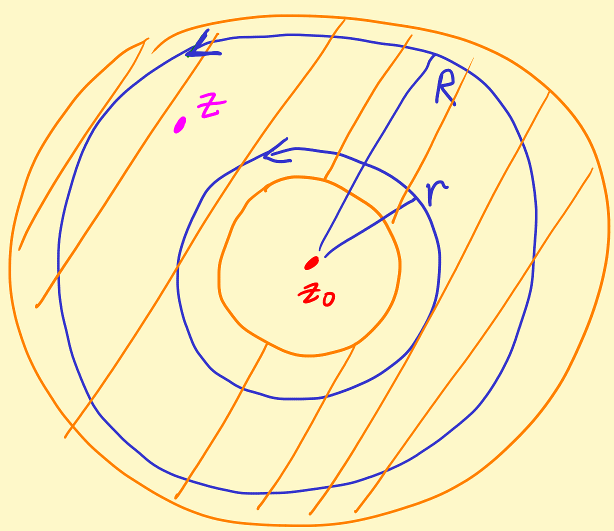
$$\frac{\cos z}{z^3} = \underbrace{\text{finitely many neg. powers}} + \text{regular power series}$$

(pole)

$$e^{1/z} = 1 + (1/z) + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots$$

infinitely many neg. powers. (essential)

<u>Laurent expansion</u> f analytic on $A(z_0; \rho_1, \rho_2)$



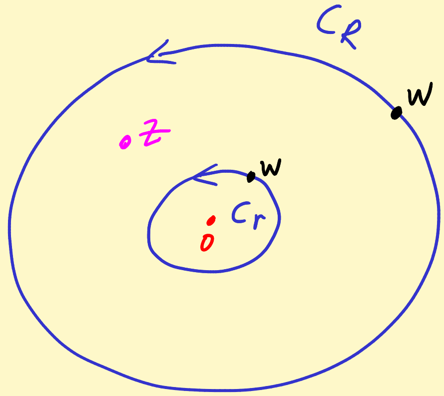
f analytic on bigger anulus

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $\quad a_n = \frac{1}{2\pi i} \int_{C_\rho(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$

where $C_\rho(z_0)$ is the counterclockwise circle of radius ρ about z_0 with $r \le \rho \le R$

Why: Take $z_0 = 0$.



Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right)\frac{f(w)}{w-z}\,dw$$

$$= \underbrace{\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w}\cdot\frac{1}{1-\frac{z}{w}}\,dw}_{\text{Get regular power series as before}} \underbrace{- \frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{z}\,\frac{1}{\frac{w}{z}-1}\,dw}_{\text{get neg. terms here}}$$

$$= \frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w}\left(1+\left(\frac{z}{w}\right)+\left(\frac{z}{w}\right)^2+\cdots\right)dw + \frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{z}\left(1+\left(\frac{w}{z}\right)+\left(\frac{w}{z}\right)^2+\cdots\right)dw$$

$$= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w^{n+1}}\,dw\right)}_{a_n} z^n + \sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{C_r} f(w)\,w^{n-1}\,dw\right)}_{a_{-n}}\frac{1}{z^n}$$

Note $\quad a_{-n} = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w^{-n+1}} \, dw \quad \checkmark$

Easy fact from Cauchy thm for an annulus.

$$\int_{C_{\rho_1}} F(w)\,dw = \int_{C_{\rho_2}} F(w)\,dw$$

when F analytic inside and on a bigger annulus

and $r \le \rho_1 \le \rho_2 \le R$.

Cauchy thm: $\left(\int_{C_{\rho_2}} - \int_{C_{\rho_1}} \right) F\,dw = 0.$

<u>Remark</u> Convention $\sum_{n=-\infty}^{\infty} a_n z^n$ converges means both

$\sum_{n=0}^{\infty} a_n z^n$ converges and $\sum_{n=1}^{\infty} a_{-n} z^{-n}$ converges.

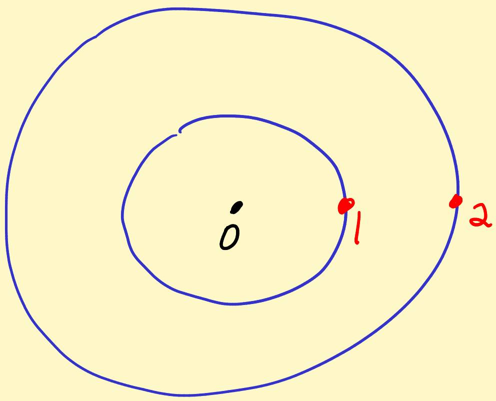
<u>Fact</u>

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \underbrace{\sum_{n=0}^{\infty} a_n z^n}_{\substack{\text{converges} \\ \text{absolutely on} \\ |z|<R}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} z^{-n}}_{\substack{\text{converges} \\ \text{absolutely on} \\ |z|>r}}$$

and the two series converge uniformly on

$A(\rho_1, \rho_2)$ when $r < \rho_1 < \rho_2 < R$.

EX Find Laurent expansion for $\dfrac{1}{(z-1)(z-2)} = \sum_{-\infty}^{\infty} a_n z^n$



Three "bands" with different series.