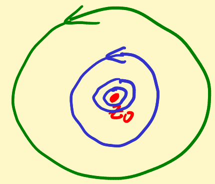
Lecture 27 Isolated singularities of analytic fcns

Def^n z_0 is called an isolated singularity of an analytic fcn $f(z)$ if there exists an $\varepsilon > 0$ such that f is analytic on $D_\varepsilon(z_0) - \{z_0\}$.

Only 3 types of isolated singularities



Know $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

converges absolutely on $D_\varepsilon(z_0) - \{z_0\}$.

3 possible cases

1) $a_n = 0$ for all negative n.

z_0 is a removable singularity.

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{regular power series with RofC } R \geq \varepsilon} \longrightarrow \text{analytic fcn on } D_\varepsilon(z_0)$$

Aha! If we set $f(z_0) = a_0$, then f becomes analytic on $D_\varepsilon(z_0)$.

EX $\frac{\sin z}{z}$ $\leftarrow$ should be $= 1$ at $z = 0$.

Facts z_0 is removable $\iff$ $\lim_{z \to z_0} f(z)$ exists

$\iff$ f is bounded on $D_\rho(z_0) - \{z_0\}$ for some ρ with $0 < \rho < \varepsilon$.

2) Only finitely many negative terms.

$$f(z) = \underbrace{\frac{a_{-N}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{z-z_0}}_{\substack{\text{rational fcn} \\ \text{called "the principal part"} \\ \text{of } f \text{ at the pole } z_0}} + \underbrace{a_0 + a_1(z-z_0) + \cdots}_{\substack{\text{regular} \\ \text{power series}}}$$

Note: Assume $a_{-N} \neq 0$ is the "first" non-zero neg coeff.

Then z_0 is called a pole of order N.

Hmmm. $$(z-z_0)^N f(z) = \underbrace{a_{-N} + a_{-N+1}(z-z_0) + \cdots + a_0(z-z_0)^N + \cdots}_{\substack{\text{regular power series, RoC} > 0 \\ \text{converges to analytic fcn } F(z) \\ \text{on } D_\varepsilon(z_0).}}$$

Note that $F(z_0) = a_{-N} \neq 0$.

$$f(z) = (z-z_0)^{-N} F(z)$$ ← very similar to a zero of order N, but with $-N$ where N was!

Important consequence: $\lim_{z \to z_0} f(z) = \infty$ ← defining feature at a pole.

EX $$\frac{\sin z}{z^4} = \frac{z - \frac{z^3}{3!} + \cdots}{z^4} = \underbrace{\frac{1}{z^3} - \frac{1/3!}{z}}_{\text{P.P.}} + \underbrace{\frac{1}{5!} z + \cdots}_{\text{entire}}$$

pole of order 3 at $z=0$.

3) ∞ many non-zero negative coeff

EX $\quad e^{1/z} = 1 + \left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots$

Shreds complex plane at $z=0$

Picard's (big) theorem Suppose f has an essential sing at z_0. Say f is analytic on $D_\varepsilon(z_0) - \{z_0\}$. For ρ with $0 < \rho < \varepsilon$, no matter how small, f maps $D_\rho(z_0) - \{z_0\}$ infinite-to-one onto $\mathbb{C}$ with one possible missing pt (like $e^{1/z}$ misses 0).

EX What kind of singularity does $\cos\frac{1}{z}$ have at $z=0$?

Hmmm. For t real, $\cos\frac{1}{t}$ is bounded by 1 in absolute value and it wiggles like hell as $t \to 0$.

So $\cos\frac{1}{z}$ couldn't possibly have a limit as $z \to 0$.

So $z=0$ is not removable.

Similarly $\cos\frac{1}{z}$ can't $\to \infty$ either.

So $z=0$ is not a pole.

$z=0$ must be essential.

Riemann removable singularity thm Suppose f is analytic and bounded on $D_\varepsilon(z_0) - \{z_0\}$. Then z_0 is removable.

Way false $\mathbb{R} \to \mathbb{R}$. $\sin \frac{1}{t}$ is bnded on $(-\varepsilon, \varepsilon) - \{0\}$.

Pf: (Book uses Picard's thm). bounded, so not a pole. bounded, so it can't plaster $\mathbb{C}$ as in Picard's. Must be removable.

[Proof without Picard's.
$$F(z) = \begin{cases} (z-z_0)^2 f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

Show F is analytic on disc about z_0.

Use power series to see $F(z)$ has a zero of order 2 at z_0. See that $f(z) =$ power series.]

Def a_{-1} in Laurent expansion is called the residue of f at z_0.

$$a_{-1} = \operatorname{Res}_{z_0} f.$$

Note
$$a_{-1} = \frac{1}{2\pi i} \int_{C_\rho(z_0)} f(z)\, dz$$

$$\left(\text{because} \qquad a_n = \frac{1}{2\pi i} \int_{C_\rho(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right)$$

Baby residue thm: $$\int_{C_\rho(z_0)} f(z)\,dz = 2\pi i \, \mathrm{Res}_{z_0} f$$

for small circles $C_\rho(z_0)$ about sing pt z_0.

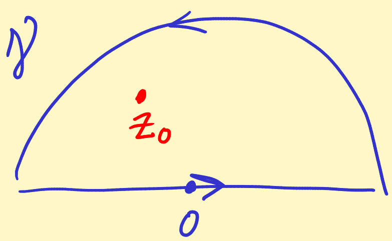
Importance of the residue

EX

residue

$$f(z) = \frac{A_N}{(z-z_0)^N} + \frac{A_{N-1}}{(z-z_0)^{N-1}} + \cdots + \frac{A_2}{(z-z_0)^2} + \frac{A_1}{(z-z_0)} + \underbrace{g(z)}_{\text{entire}}$$

$$\frac{d}{dz}\left[A_N \frac{(z-z_0)^{-N+1}}{(-N+1)}\right] \qquad\qquad \frac{d}{dz}\left[\frac{-A_2}{(z-z_0)}\right]$$

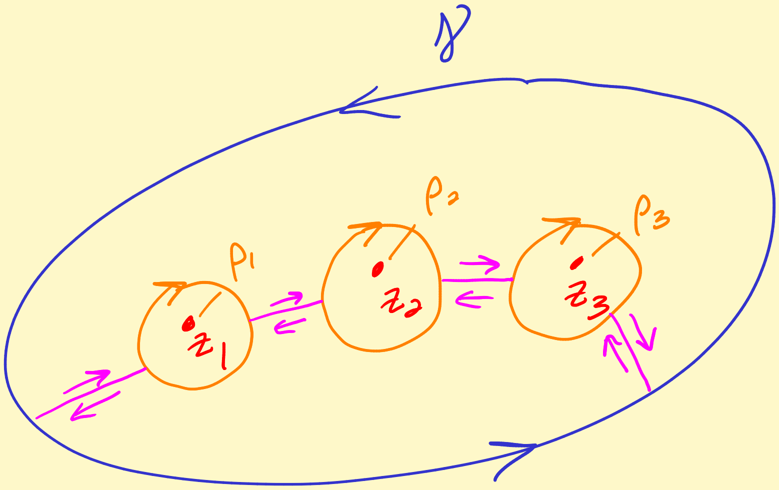


$$\int_\gamma f\,dz = \underbrace{0+0+\cdots+0}_{\text{Fund Thm calc}} + \underbrace{\int_\gamma \frac{A_1}{z-z_0}\,dz}_{=2\pi i A_1} + \underbrace{0}_{\text{Cauchy thm}}$$

Cauchy integral formula using $f(z) \equiv A_1$.

$$= 2\pi i \, \mathrm{Res}_{z_0} f \, .$$

Big idea



f analytic inside and on γ with singularities at z_j's

$$0 = \left(\int_{top} + \int_{bottom} \right) f \, dz = \int_\gamma f \, dz - \sum_{j=1}^{3} \underbrace{\int_{C_{\rho_j}(z_j)} f \, dz}_{2\pi i \, Res_{z_j} f}$$

So $$\int_\gamma f \, dz = 2\pi i \sum_{j=1}^{3} Res_{z_j} f$$