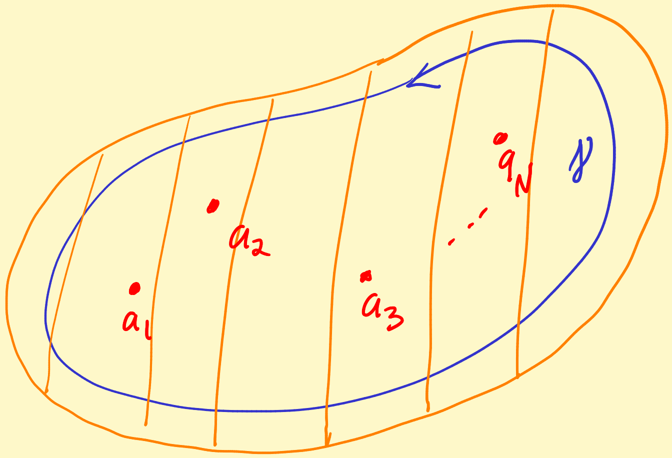
Lecture 28 The residue theorem

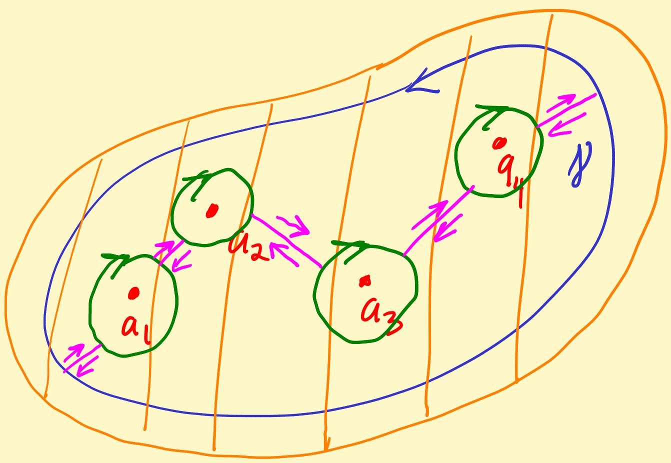
γ simple closed curve.



f analytic inside and on γ except at finitely many isolated singularities inside γ. $\{a_n\}_{n=1}^{N}$

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{n=1}^{N} \mathrm{Res}_{a_n} f$$

Pf



$$\underbrace{\Big(\int_{\gamma_{TOP}} + \int_{\gamma_{BOTTOM}}\Big)}_{\substack{0 \;+\; 0 \\ \text{by Cauchy's thm}}} f\,dz = \int_\gamma f\,dz + \sum_{n=1}^{N} \underbrace{\int_{-C_{\varepsilon_n}(a_n)} f\,dz}$$

$$-\int_{\underbrace{C_{\varepsilon_n}(a_n)}_{\text{counterclockwise}}} f\,dz = -2\pi i\, \underbrace{\mathrm{Res}_{a_n} f}_{\substack{= a_{-1} \\ \text{term in} \\ \text{Laurent expansion} \\ \text{for } f \text{ at } a_n}}$$

Applications of the residue thm

Famous integrals : Fourier transform

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-isx} f(x)\, dx$$

Big fact:

$$\widehat{f'}(s) = -is\,\hat{f}(s)$$

$$\widehat{f''}(s) = -s^2\,\hat{f}(s)$$

(Fourier transf) : {Diff eqns} $\longrightarrow$ $\underbrace{\{\text{algebra}\}}_{\text{solve for } \hat{f}}$

Bigger fact: Can undo the F.T.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\,\hat{f}(s)\, ds$$

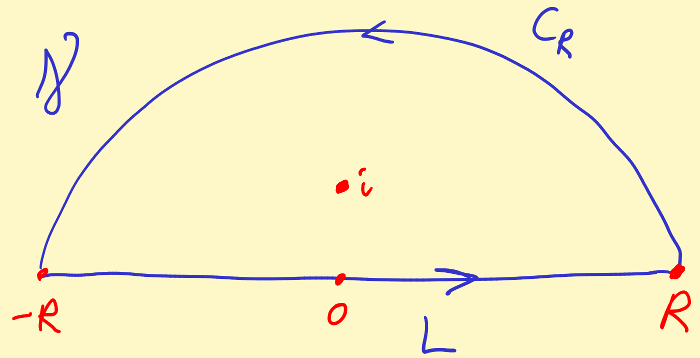
EX Calcutate

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+1}\, dx$$

$\hat{g}(-s)$ for $g(x) = \frac{1}{x^2+1}$

when $s > 0$.

$$f(z) = \frac{e^{isz}}{z^2+1}$$



$$\int_L f(z)\, dz = \int_{-R}^{R} \frac{e^{ist}}{t^2+1}\cdot 1\, dt$$

$L: z(t) = t,\ -R \le t \le R$

$z'(t) = 1$

$$= \int_{-R}^{R} \underbrace{\frac{\cos st}{t^2+1}}_{\text{even}} \, dt + \underbrace{i \int_{-R}^{R} \frac{\sin st}{\underbrace{t^2+1}_{\text{odd}}} \, dt}_{=0}$$

$$= 2\int_{0}^{R} \frac{\cos st}{t^2+1} \, dt \quad \longleftarrow \text{want, as } R \to \infty.$$

Res thm $\quad \int_{\gamma} f \, dz = 2\pi i \, \mathrm{Res}_i f$

<u>Claim</u> $\quad \int_{C_R} f \, dz \longrightarrow 0$

1) e^{isz} is bounded in UHP $\longleftarrow$ Upper Half Plane

Why: $\left| e^{is(x+iy)} \right| = \left| e^{isx} e^{-sy} \right| = \underbrace{\left| e^{isx} \right|}_{=1} \underbrace{\left| e^{-sy} \right|}_{e^{-sy}}$

$$= e^{-sy} \leq e^{0} = 1 \quad \text{when } y \geq 0$$

$$(\text{because } s > 0).$$

2) Denominator estimate $\quad |z^2+1| \geq \underbrace{\big| |z^2| - |1| \big|}$

on C_R: $\quad \underbrace{|R^2 - 1|}_{= R^2 - 1 \text{ when } R > 1.}$

So $\left| \frac{e^{isz}}{z^2+1} \right| \le \frac{1}{R^2-1}$ on C_R when $R>1$.

Basic integral estimate: $\left| \int_{C_R} \frac{e^{isz}}{z^2+1}\,dz \right| \le \max_{C_R} |f(z)| \cdot \text{Length}(C_R)$

$$\le \frac{1}{R^2-1} \cdot (\pi R)$$

(when $R>1$) $\longrightarrow 0$ as $R \to \infty$. ✓

Last step. $\frac{e^{isz}}{z^2+1} = \frac{e^{isz}}{(z-i)(z+i)}$

$$= \frac{1}{(z-i)} \left[\underbrace{\frac{e^{isz}}{z+i}}_{A_0 + A_1(z-i) + A_2(z-i)^2 + \cdots} \right]$$

analytic near i

Res f

$$= \underbrace{\frac{A_0}{z-i}}_{\text{principal part}} + \underbrace{A_1 + A_2(z-i) + \cdots}_{\text{regular power series}}$$

Taylor's formula $A_0 = \left[\frac{e^{isz}}{z+i} \right] \Big|_{z=i} = \frac{e^{isi}}{i+i} = \frac{e^{-s}}{2i}$

Done!

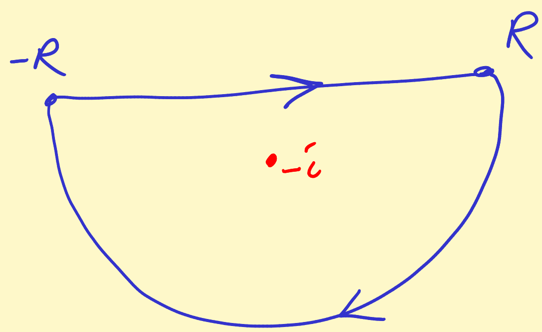
$$\left(\int_L + \int_{C_R}\right) f \, dz = 2\pi i \operatorname{Res}_i f$$

$$\int_{-\infty}^{\infty} \frac{e^{ist}}{t^2+1} dt + 0 = 2\pi i \left(\frac{e^{-s}}{2i}\right)$$

$$2\int_0^{\infty} \frac{\cos st}{t^2+1} dt = \pi e^{-s} \quad \text{when } s>0.$$

What if $s<0$?

$|e^{isz}| \le 1$ in LHP.



$-2\pi i \operatorname{Res}_{-i} f(z)$

clockwise!

Important formula Suppose f and g are analytic on $D_r(a)$ and g has a simple zero at a (i.e., $g(a)=0$, but $g'(a) \ne 0$. zero of order 1).

Then $$\operatorname{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}$$

Why $g(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$

$a_0 = g(a) = 0 \qquad a_1 = \frac{g'(a)}{1!} \neq 0$

$$= a_1(z-a) + a_2(z-a)^2 + \cdots$$

$$= (z-a)\Big[\underbrace{a_1 + a_2(z-a) + \cdots}_{G(z) \text{ analytic}}\Big]$$

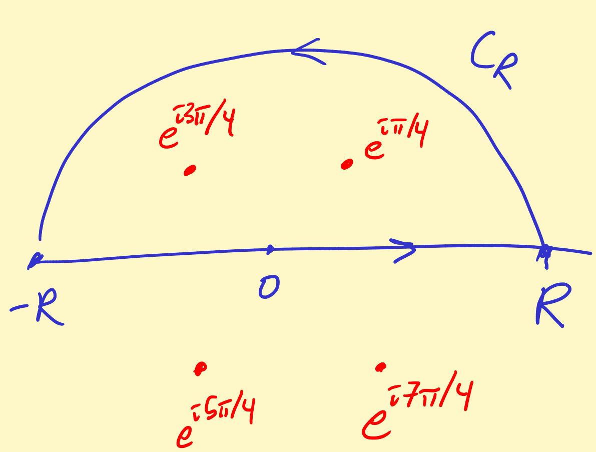
Note: $G(a) = a_1 = g'(a)$

Now $$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-a)G(z)} = \frac{1}{z-a}\Bigg[\underbrace{\frac{f(z)}{G(z)}}_{A_0 + A_1(z-a) + \cdots}\Bigg]$$

$$= \frac{\overset{\text{Res } a}{A_0}}{z-a} + \underbrace{A_1 + A_2(z-a) + \cdots}_{\text{analytic near } a}$$

$$\text{Res}_a \frac{f}{g} = A_0 = \left.\frac{f(z)}{G(z)}\right|_{z=a} = \frac{f(a)}{g'(a)} \quad \checkmark$$

EX $$I = \int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$$



$$f(z) = \frac{1}{z^4+1}$$

$$\int_L f\,dz + \int_{C_R} f\,dz = 2\pi i\left(\mathrm{Res}_{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i} f + \mathrm{Res}_{-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}} f\right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$I \quad + \quad 0$$

$$\frac{1}{z^4+1} = \frac{f(z)}{g(z)} \qquad \begin{matrix} \leftarrow f(z) \equiv 1 \\ \leftarrow g(z) = z^4+1 \end{matrix}$$

$$g(e^{i\pi/4}) = 0$$

$$g'(z) = 4z^3 \qquad g'(e^{i\pi/4}) = 4(e^{i\pi/4})^3 = 4e^{i3\pi/4} \neq 0$$

$$\mathrm{Res}_{e^{i\pi/4}} \frac{f}{g} = \frac{1}{4e^{i3\pi/4}} = \frac{1}{4} e^{-i3\pi/4} = \frac{1}{4}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)$$

$$\mathrm{Res}_{e^{i3\pi/4}} \frac{1}{z^4+1} = \frac{1}{4(e^{i3\pi/4})^3} = \frac{1}{4} e^{-i9\pi/4} = \frac{1}{4}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i \cdot \frac{1}{4}\left[-i\sqrt{2}\right] = \frac{\pi}{\sqrt{2}}$$