Lecture 29 More applications of the residue theorem

HWK 8 due Thurs

Useful formula: $\operatorname{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}$ when f, g analytic near a and g has a simple zero at a.

$g(a)=0, \; g'(a)\neq 0.$

Ex
$$\operatorname{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} = \frac{1}{4z^3}\Bigg|_{z=e^{i\pi/4}} = \frac{1}{4e^{i3\pi/4}} = \frac{e^{-i3\pi/4}}{4}$$

Book:
$$\frac{1}{z^4+1} = \frac{1}{(z-e^{i\pi/4})(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})}$$

$$= \frac{1}{z-e^{i\pi/4}} \cdot \Bigg[\underbrace{\frac{1}{(\quad)(\quad)(\quad)}}_{A_0+A_1(z-a)+\cdots}\Bigg]$$

$a=e^{i\pi/4}$

analytic near $e^{i\pi/4}$ and $\neq 0$

$$= \underbrace{\frac{A_0}{z-a}}_{\text{princ. part}} + \underbrace{A_1 + A_2(z-a)+\cdots}_{\text{regular power series}}$$

Res_a

$$\operatorname{Res}_a \frac{1}{z^4+1} = A_0 = \left[\frac{1}{(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})}\right]\Bigg|_{z=e^{i\pi/4}}$$

Ugh!

Theorem $P(x)$ and $Q(x)$ polynomials,

$$\deg Q \geq \deg P + 2$$

and Q has no zeroes on $\mathbb{R}$.

Then $$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a \in \mathfrak{A}} \mathrm{Res}_a \frac{P(z)}{Q(z)}$$

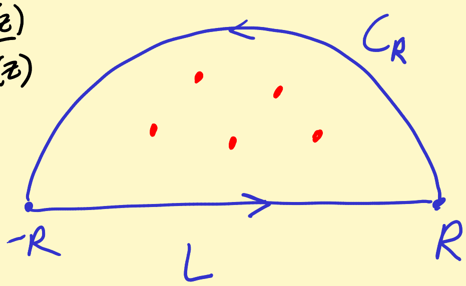
where $\mathfrak{A} = \{ z : Q(z) = 0, \underbrace{\mathrm{Im}\, z > 0}_{\text{UHP}} \}$

Pf: Key: Basic estimate for polynomials

$$a|z|^{\deg P} \le \underline{|P(z)| \le A|z|^{\deg P}}, \quad |z| > R_P$$

$$\underline{b|z|^{\deg Q} \le |Q(z)|} \le B|z|^{\deg Q}, \quad |z| > R_Q$$

$f(z) = \frac{P(z)}{Q(z)}$



$\mathfrak{A}$ inside the D

Key: $$\left| \int_{C_R} \frac{P}{Q}\,dz \right| \le \left(\max_{C_R} \left| \frac{P(z)}{Q(z)} \right| \right) \mathrm{Length}(C_R)$$

$$\le \underbrace{\frac{A R^{\deg P}}{b R^{\deg Q}} \cdot (\pi R)}_{\frac{\pi A}{b} \frac{1}{R^{\deg Q - \deg P - 1}}} \quad \text{when } R > \max(R_P, R_Q)$$

$\leftarrow$ exponent ≥ 1

$\longrightarrow 0$ as $R \to \infty$.

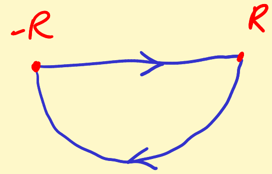
Residue thm: $$\left(\int_L + \int_{C_R}\right) f\,dz = 2\pi i \sum_{a\in\mathcal{U}} \operatorname{Res}_a f \quad ✓$$

$$R\to\infty: \quad I + 0$$

<u>Remark</u> $|e^{isz}| \le 1$ in UHP when $s \ge 0$

$|e^{isz}| \le 1$ in LHP when $s < 0$

<u>EX</u>: $$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{isx}\,dx = 2\pi i \sum_{a\in\mathcal{U}} e^{isz}\frac{P(z)}{Q(z)} \qquad (s>0)$$

$s<0$ case: Do 

clockwise!

. $I = -2\pi i \sum_{a\in\mathcal{U}^-}$ zeroes of Q in LHP.

<u>Trig integrals</u> $$I = \int_0^{2\pi} \cos^6 t\,dt$$

$$\cos t = \frac{e^{it}+e^{-it}}{2} = \frac{e^{it}+\frac{1}{e^{it}}}{2} = \frac{z+\frac{1}{z}}{2}$$

C: $z(t) = e^{it}$

$z'(t) = ie^{it}$

$dz = ie^{it}\,dt$

$$I = \int_0^{2\pi}\left[\frac{e^{it}+\frac{1}{e^{it}}}{2}\right]^6 \cdot \frac{ie^{it}\,dt}{\underbrace{ie^{it}}_{iz(t)}} \quad dz$$

$$= \int_C \left[\frac{z+\frac{1}{z}}{2}\right]^6 \cdot \frac{1}{iz} \cdot dz$$

$$= 2\pi i \operatorname{Res}_0 \frac{1}{iz}\left(\frac{z+\frac{1}{z}}{2}\right)^6$$

$$\frac{1}{iz}\left(\frac{z+z^{-1}}{2}\right)^6 = \frac{1}{2^6}\left(z^6+6z^4+15z^2+20+\frac{15}{z^2}+\frac{6}{z^4}+\frac{1}{z^6}\right)\frac{1}{iz}$$

$$= \cdots + \frac{\left(\frac{20}{i2^6}\right)}{z} + \cdots$$

Res!

$$I = 2\pi i \cdot \frac{20}{i2^6} = \frac{5\pi}{8}$$

<u>General method</u> $\quad I = \int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

R rational fcn

$$I = \int_C \underbrace{R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)}_{\text{need no singularities on } C} \cdot \frac{1}{iz}\, dz$$

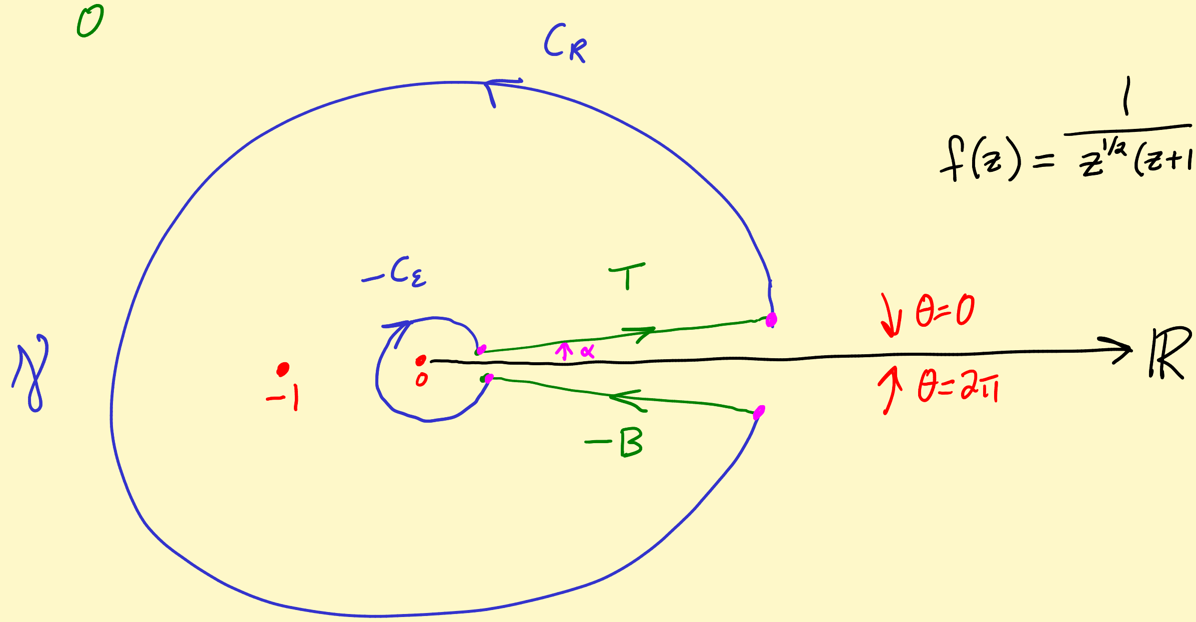
$$= 2\pi i \sum_{a \in \Omega} \operatorname{Res}_a f$$

where Ω are singular pts of $f(z) = \frac{1}{iz} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)$ inside unit circle C.

<u>Integrals involving complex log fcns</u>

$$I = \int_0^\infty \frac{1}{\sqrt{x}\,(x+1)}\,dx$$

$$f(z) = \frac{1}{z^{1/2}(z+1)}$$



Define branch $z^{1/2} = e^{\frac{1}{2}\log_0 z}$ where $\log_0 z = \operatorname{Ln}|z| + i\theta$

where $\theta \in \arg z$, $0 < \theta < 2\pi$

$$\int_\gamma f(z)\,dz = 2\pi i \ \operatorname{Res}_{-1} f(z)$$

$$\operatorname{Res}_{-1} f(z) = \operatorname{Res}_{-1} \frac{1}{(z-(-1))}\cdot\frac{1}{z^{1/2}}$$

$$= \frac{1}{(-1)^{1/2}} = \frac{1}{e^{\frac{1}{2}\log_0(-1)}}$$

$$= \frac{1}{e^{\frac{1}{2}(\operatorname{Ln}|-1| + i\pi)}} = \frac{1}{e^{i\pi/2}} = \frac{1}{i}$$

Easy $\int_{C_R}$ and $\int_{C_\varepsilon} \to 0$ as $R \to \infty$, $\varepsilon \to 0$.

Fact $\left| \left(z^{1/2} \right) \right| = |z|^{1/2}$

$$\left| e^{\frac{1}{2}\log z} \right| = \left| e^{\frac{1}{2}\left(\ln|z| + i\theta\right)} \right| = \left| e^{\frac{1}{2}\ln|z|} \cdot e^{i\theta/2} \right|$$

$$= \underbrace{e^{\frac{1}{2}\ln|z|}}_{|z|^{1/2}} \cdot 1 \leftarrow 1 = \left| e^{i\theta/2} \right|$$

does not depend on branch!

✓

$$\left| \int_{C_R} f(z)\, dz \right| \le \underbrace{\max_{C_R} \left| \frac{1}{z^{1/2}(z+1)} \right|}_{\frac{1}{R^{1/2}} \cdot \frac{1}{R-1}} \cdot (2\pi R)$$

when $R > 1$

$\to 0$ as $R \to \infty$.

Do the $\left| \int_{C_\varepsilon} \right|$ for next time.

$$|z+1| \ge \big|\, \underbrace{|z|}_{\substack{R \\ \varepsilon}} - \underbrace{|1|}_{1} \,\big|$$