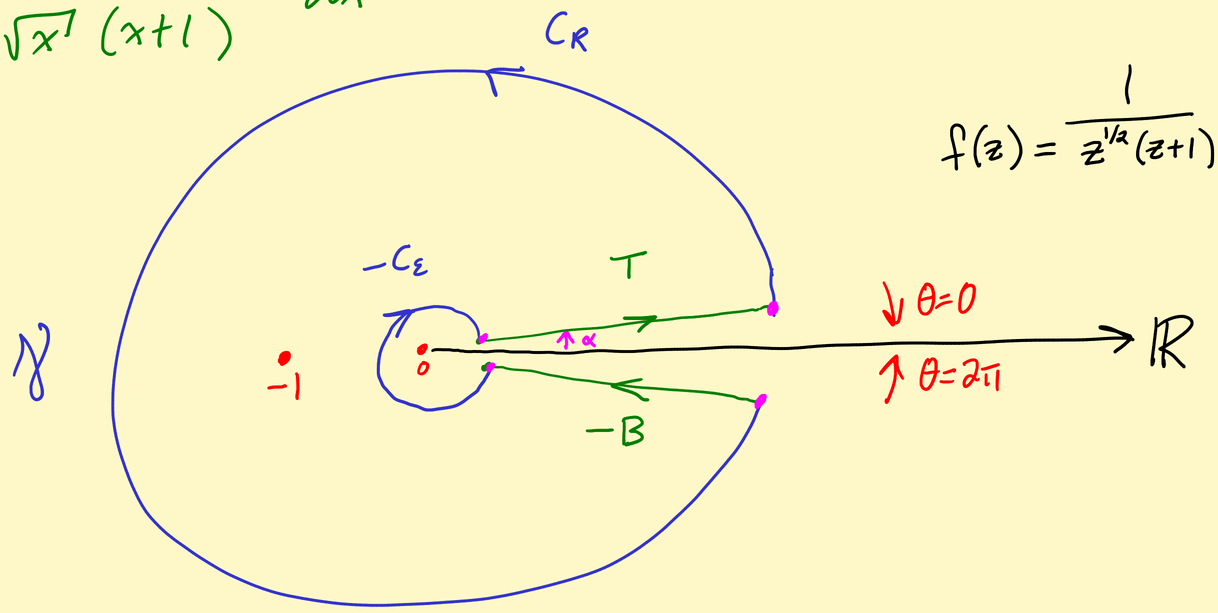
Lecture 30 More residue theorem examples

HWK 9 due Thurs

$$I = \int_0^\infty \frac{1}{\sqrt{x}\,(x+1)}\, dx$$

$$f(z) = \frac{1}{z^{1/2}(z+1)}$$



Define branch $z^{1/2} = e^{\frac{1}{2}\log_0 z}$ where $\log_0 z = \mathrm{Ln}|z| + i\theta$

where $\theta \in \arg z$, $0 < \theta < 2\pi$

$$\int_\gamma f(z)\, dz = 2\pi i \ \mathrm{Res}_{-1} f(z)$$

$$\mathrm{Res}_{-1} f(z) = \mathrm{Res}_{-1} \frac{1}{(z-(-1))} \cdot \frac{1}{z^{1/2}}$$

$$= \frac{1}{(-1)^{1/2}} = \frac{1}{e^{\frac{1}{2}\log_0(-1)}}$$

$$= \frac{1}{e^{\frac{1}{2}(\mathrm{Ln}|-1| + i\pi)}} = \frac{1}{e^{i\pi/2}} = \frac{1}{i}$$

$\int_{C_R}$ and $\int_{C_\varepsilon} \to 0$ as $R \to \infty$, $\varepsilon \to 0$:

Fact $\left|\left(z^{1/2}\right)\right| = |z|^{1/2}$

$$\left[\begin{array}{ll} & w^2 = z \\ \underline{\underline{OR}} & \underbrace{|w^2|}_{|w|^2} = |z| \qquad |w| = |z|^{1/2} \end{array}\right.$$

$$\left|e^{\frac{1}{2}\log z}\right| = \left|e^{\frac{1}{2}(\ln|z| + i\theta)}\right| = \left|e^{\frac{1}{2}\ln|z|} \cdot e^{i\theta/2}\right|$$

$$= \underbrace{e^{\frac{1}{2}\ln|z|}}_{|z|^{1/2}} \cdot 1 \leftarrow 1 = |e^{i\theta/2}|$$

does not depend on branch!

✓

$$\left|\int_{C_R} f(z)\,dz\right| \le \underbrace{\underset{C_R}{Max}\left|\frac{1}{z^{1/2}(z+1)}\right|}_{\le 1} \cdot \underbrace{(2\pi R)}_{L < 2\pi R}$$

$$\frac{1}{R^{1/2}} \cdot \frac{1}{R-1} \cdot (2\pi R) \qquad \text{when } R > 1$$

$\to 0$ as $R \to \infty$.

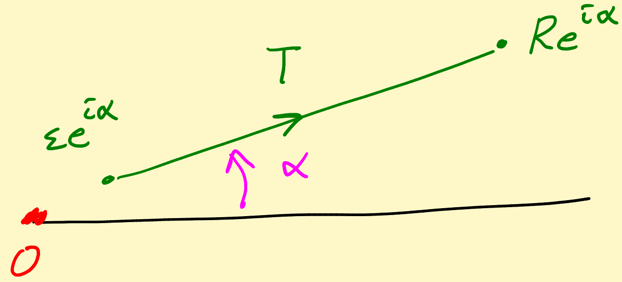
denom. est.

$$|z+1| \ge \left|\underbrace{|z|}_{R,\ \varepsilon} - \underbrace{|1|}_{1}\right| = \begin{cases} R-1 & \text{when } R>1 \\ 1-\varepsilon & \text{when } 0<\varepsilon<1 \end{cases}$$

$$\left|\int_{-C_\varepsilon} f\,dz\right| \le \frac{1}{\varepsilon^{1/2}(1-\varepsilon)} \cdot (2\pi\varepsilon) \qquad \frac{2\pi\,\varepsilon^{1/2}}{1-\varepsilon} \qquad \text{when } 0<\varepsilon<1$$

$\longrightarrow 0$ as $\varepsilon \to 0$.

<u>Next</u> $T: z(t) = ?$



$z(t) = t e^{i\alpha}, \quad \varepsilon \le t \le R$

$z'(t) = e^{i\alpha}$

$$\int_T f(z)\,dz = \int_\varepsilon^R \frac{1}{(te^{i\alpha})^{1/2}(te^{i\alpha}+1)} \cdot \underbrace{\left(e^{i\alpha}\,dt\right)}_{z'(t)dt}$$

$$= e^{i\alpha}\int_\varepsilon^R \frac{1}{\underbrace{e^{\frac{1}{2}(\ln|te^{i\alpha}| + i\alpha)}}_{\underbrace{e^{\frac{1}{2}\ln t}}_{\sqrt{t}}\; e^{i\alpha/2}}(te^{i\alpha}+1)}\,dt$$

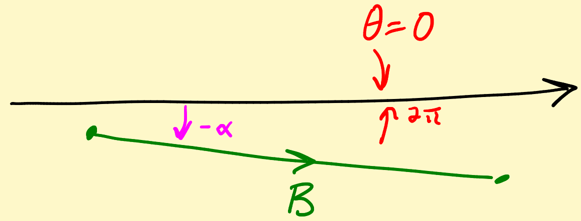
Aha! Let $\alpha \downarrow 0$. Then $e^{i\alpha} \to 1$ as $\alpha \to 0$.

$e^{i\alpha/2} \to 1$ too!

See $$\int_T f\,dz \to \int_\varepsilon^R \frac{1}{\sqrt{t}(t+1)}\,dt \longleftarrow \text{Want!}$$

Now $B: z(t) = t e^{-i\alpha}, \quad \varepsilon \le t \le R$

$$\int_B f\,dz = e^{-i\alpha}\int_\varepsilon^R \frac{1}{e^{\frac{1}{2}(\ln t + i(2\pi-\alpha))}(te^{-i\alpha}+1)}\,dt$$



As $\alpha \to 0$: $$\int_B f\,dz \to \int_\varepsilon^R \frac{1}{\sqrt{t}\,\underbrace{e^{i\frac{1}{2}\cdot 2\pi}}_{e^{i\pi}=-1}\,(t+1)}\,dt = -\int_\varepsilon^R \frac{1}{\sqrt{t}\,(t+1)}\,dt$$

$$\int_{-B} = -\int_B$$

So $$\lim_{\alpha\to 0} \int_{-B} f\,dz = \int_\varepsilon^R \frac{1}{\sqrt{t}\,(t+1)}\,dt \leftarrow \text{Want!}$$

Finally,

$$\Big(\underbrace{\int_{C_R}}_{\downarrow\, 0} + \underbrace{\int_{-B}}_{\downarrow\, \text{Want}} + \underbrace{\int_{C_{-\varepsilon}}}_{\downarrow\, 0} + \underbrace{\int_T}_{\downarrow\, \text{Want}}\Big) f\,dz = 2\pi i\left(\frac{1}{i}\right)$$

Step 1. Let $\alpha \to 0$.

$$\left(\int_{C_R} + 2\int_\varepsilon^R \frac{1}{\sqrt{t}\,(t+1)}\,dt + \int_{-C_\varepsilon}\right) f\,dz = 2\pi$$

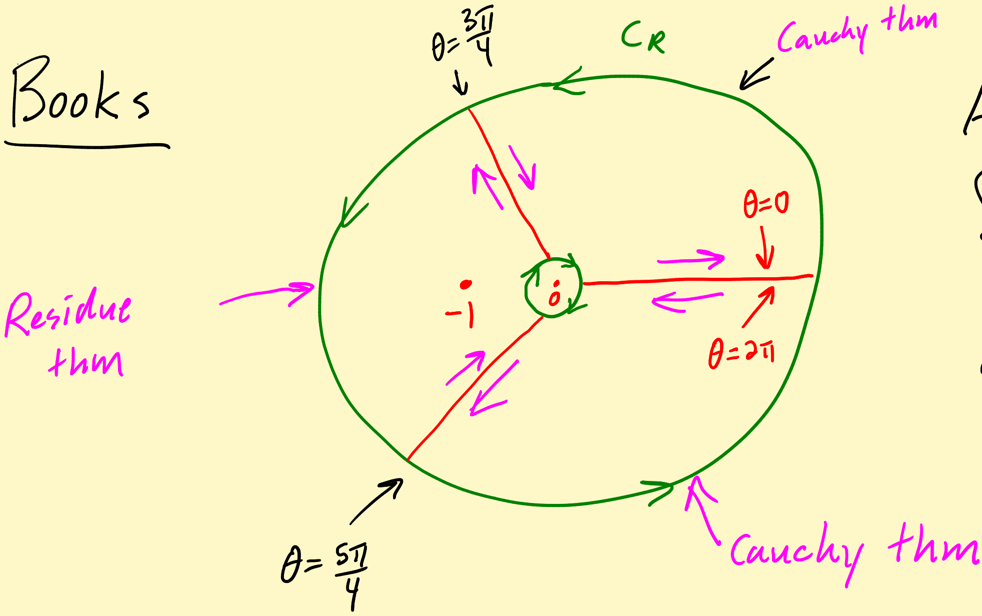
Step 2. Now let $\varepsilon \searrow 0$.

Step 3. Let $R \nearrow \infty$.

$$0 + 2I + 0 = 2\pi$$

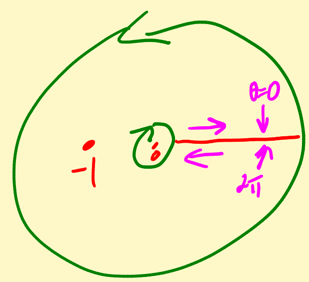
So $\quad I = \int_0^\infty \frac{1}{\sqrt{t}\,(t+1)}\, dt = \pi$

<u>Books</u>



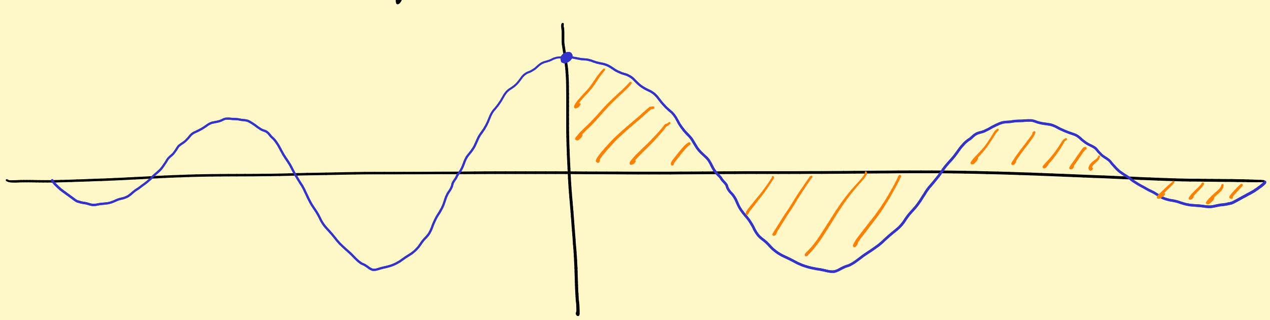
Add up counterclockwise $\int$'s along simple closed curves. Use carefully chosen branches of log

Add up. Get "residue" thm



<u>Famous integral</u> $\quad \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$

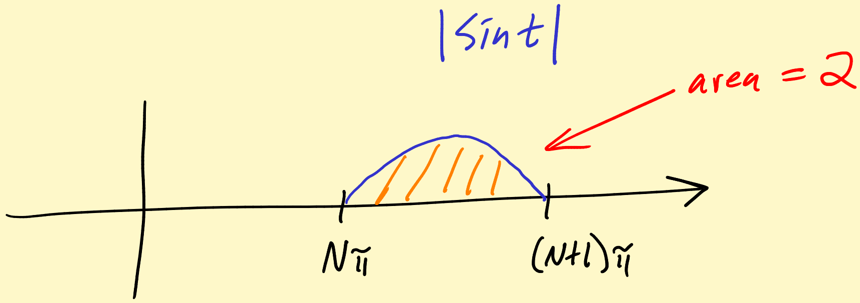
<u>Not</u> absolutely integrable, but conditionally convergent.



Alternating series test. Areas of humps decreasing and $\searrow 0$, alternating sign.

$$\lim_{R\to\infty} \int_0^R \frac{\sin t}{t}\,dt \quad \text{exists.}$$

But



$$N\pi \leq t \leq (N+1)\pi$$

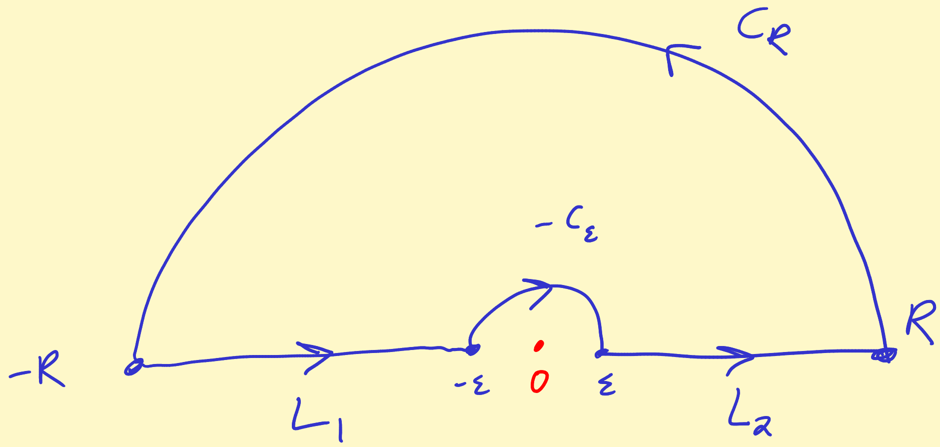
$$\boxed{\frac{|\sin t|}{(N+1)\pi} \leq \left|\frac{\sin t}{t}\right| \leq \frac{|\sin t|}{N\pi}} \quad \text{on interval}$$

$$\int_{N\pi}^{(N+1)\pi}: \quad \underbrace{\frac{2}{(N+1)\pi} \leq \int_{N\pi}^{(N+1)\pi} \left|\frac{\sin t}{t}\right| dt}_{\int_0^R \left|\frac{\sin t}{t}\right| dt \;\to\; \infty \text{ as } R\to\infty} \leq \frac{2}{N\pi}$$

because $\sum_{N=1}^{\infty} \frac{1}{N} = \infty$

rate is same as divergence of "harmonic series"

Plan



$$f(z) = \frac{e^{iz}}{z}$$