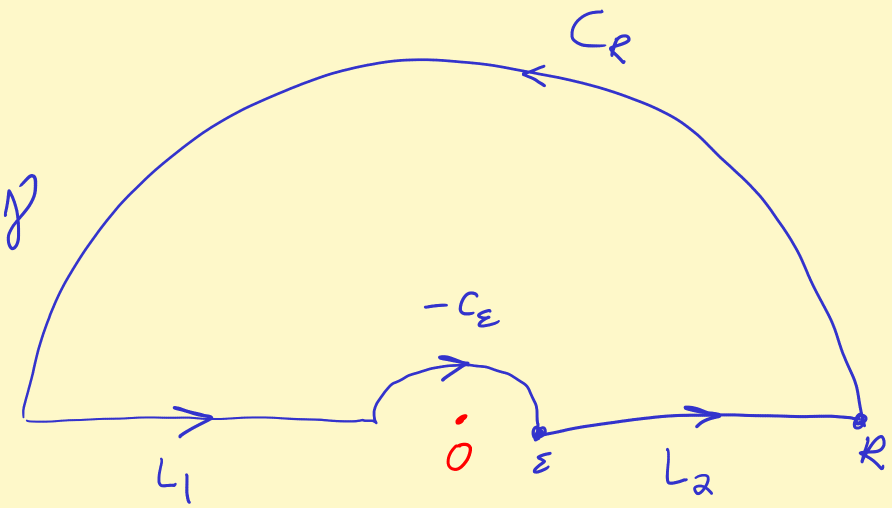
Lecture 31 Jordan's lemma

$\int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt = \pi$



$f(z) = \frac{e^{iz}}{z}$

Plan $\quad L_j : z(t) = t, \quad \varepsilon \le t \le R, \; -R \le t \le -\varepsilon, \quad z'(t) = 1$

$$\left(\int_{L_1} + \int_{L_2}\right) \frac{e^{iz}}{z}\, dz = \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \frac{\overbrace{\cos t + i \sin t}^{e^{it}}}{t} \cdot 1 \cdot dt$$

$\frac{\cos t}{t}$	$\frac{\sin t}{t}$
odd	even

$$= 0 + 2i \int_{\varepsilon}^{R} \frac{\sin t}{t}\, dt \longleftarrow \text{Want!}$$

2) $\int_\gamma f\, dz = 0$ by Cauchy's thm.

3) $\int_{C_R} f\, dz \to 0$ as $R \to \infty$ by Jordan's lemma.

4) $$\int_{-C_\varepsilon} \frac{e^{iz}}{z}\, dz \xrightarrow{\varepsilon \to 0} \underset{\substack{\uparrow \\ \text{clockwise}}}{-} (2\pi i) \left[\operatorname{Res}_0 \frac{e^{iz}}{z}\right] \cdot \underset{\substack{\uparrow \\ \text{half circle}}}{\left(\frac{1}{2}\right)}$$

$= -\pi i$

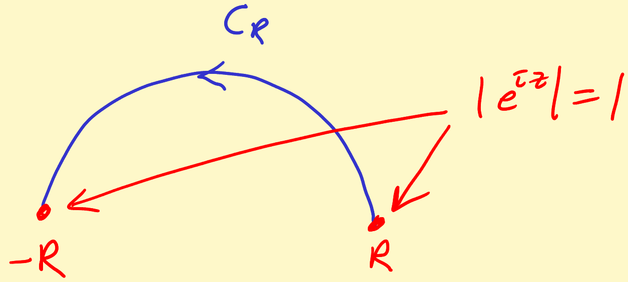
How to remember 4: Whole circle $-C_\varepsilon(0)$

$$\int_{-C_\varepsilon(0)} \frac{e^{iz}}{z}\,dz = -(2\pi i)\,\underbrace{\operatorname{Res}_0 \frac{e^{iz}}{z}}_{\frac{e^{i\cdot 0}}{1} = 1} = -2\pi i$$

In limit as $\varepsilon \downarrow 0$ for <u>half</u> circle, get $\frac{1}{2}$ of this.

3) $$\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| \le \left(\max_{C_R} \frac{|e^{iz}|}{R}\right)(\underbrace{\pi R}_{\text{length}(C_R)})$$

$$\left|e^{i(x+iy)}\right| = \left|e^{ix}e^{-y}\right| = \underbrace{\left|e^{ix}\right|}_{1}\cdot\underbrace{\left|e^{-y}\right|}_{e^{-y}} = e^{-y} \le 1$$



Ouch! Basic est only yields $\left|\int_{C_R}\right| \le \underbrace{\frac{1}{R}\cdot(\pi R)}_{\pi}$

Doesn't show that $\int_{C_R} \to 0$.

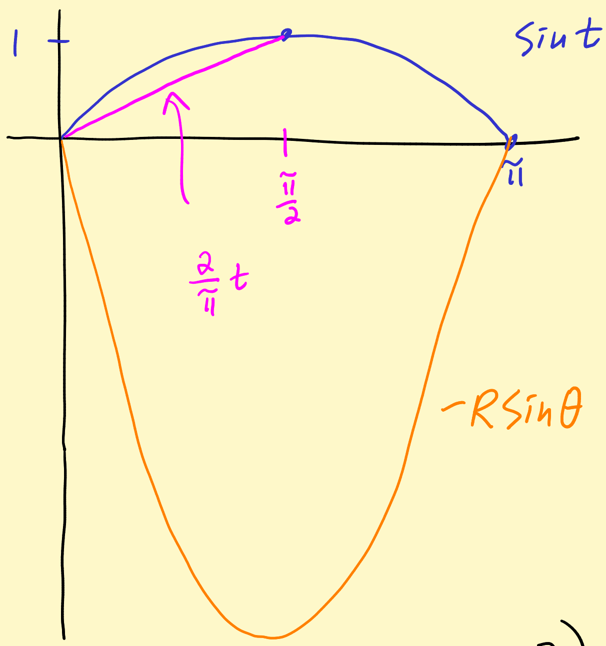
Need Jordan's lemma: $C_R: z(t) = Re^{it},\ 0 \le t \le \pi$.
$z'(t) = iRe^{it}$

$$\int_{C_R} \frac{e^{iz}}{z}\, dz = \int_0^{\pi} \frac{e^{i(Re^{it})}}{Re^{it}}\, iRe^{it}\, dt$$

$$= i\int_0^{\pi} \underbrace{e^{i(R\cos t + iR\sin t)}}_{e^{iR\cos t}\, e^{-R\sin t}}\, dt$$

So
$$\left|\int_{C_R} \frac{e^{iz}}{z}\, dz\right| \le 1\cdot\int_0^{\pi} \underbrace{\left|e^{iR\cos t}\right|}_{=1}\cdot\underbrace{\left|e^{-R\sin t}\right|}_{e^{-R\sin t}}\, dt$$

$$\le \int_0^{\pi} e^{-R\sin t}\, dt$$



Notice: 1) Symmetric about $t = \frac{\pi}{2}$

2) $0 \le \frac{2t}{\pi} \le \sin t$ on $[0, \frac{\pi}{2}]$

So $\boxed{-R\sin t \le -R\frac{2}{\pi}t \le 0}$ on $[0, \frac{\pi}{2}]$

3) e^x is increasing. So

$$e^{-R\sin t} \le e^{-\frac{2R}{\pi}t} \le \underbrace{e^0}_{1}$$

So
$$\int_0^{\pi} e^{-R\sin t}\, dt \overset{\text{symm.}}{=} 2\int_0^{\pi/2} e^{-R\sin t}\, dt$$

$$\le \underbrace{2\int_0^{\pi/2} e^{-\frac{2R}{\pi}t}\,dt}_{}$$

$$2\left[-\frac{\pi}{2R}e^{-\frac{2R}{\pi}t}\right]_0^{\pi/2}$$

$$\frac{\pi}{R}\left[-e^{-R}-(-1)\right]$$

$$\frac{\pi}{R}\left(1-e^{-R}\right) \le \frac{\pi}{R}\cdot 1 \longrightarrow 0$$

as $R \to \infty$.

$0 \le e^{-R} \le 1$

4) On $-C_\varepsilon$, $\quad C_\varepsilon$: $z(t) = \varepsilon e^{it}$, $0 \le t \le \pi$

$$\frac{e^{iz}}{z} = \frac{1 + (iz) + \frac{1}{2!}(iz)^2 + \cdots}{z}$$

$$= \frac{1}{z} + \underbrace{\overbrace{\left(i - \frac{1}{2!}z - \frac{i}{3!}z^2 + \cdots\right)}^{H(z)\text{ entire fcn!}}}_{\substack{\text{converges for same } z \\ \text{that series for } e^{iz} \text{ does!} \\ \text{So RoC} = \infty.}}$$

Res

$$\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz = -\underbrace{\int_{C_\varepsilon} \frac{1}{z}\,dz}_{} - \underbrace{\int_{C_\varepsilon} H(z)\,dz}_{\text{claim: } \to 0 \text{ as } \varepsilon \to 0.}$$

$$-\int_0^{\pi} \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\, dt$$

$$-i\int_0^{\pi} dt = -\pi i$$

Done when I show $\left|\int_{C_\varepsilon} H\, dz\right| \to 0$

$$\le \underbrace{\left(\max_{C_\varepsilon} |H|\right)}(\pi\varepsilon) \to 0 \text{ as } \varepsilon \searrow 0.$$

$|H|$ continuous on $\overline{D_1(0)}$, so it attains a max M there. So $\max_{C_\varepsilon} |H| \le M$ when $\varepsilon \le 1$.

Finally $$\Bigg[\underbrace{\left(\int_{L_1} + \int_{L_2}\right)}_{\downarrow\; 2i\int_0^\infty \frac{\sin t}{t}\,dt} + \Bigg(\underbrace{\int_{C_R}}_{\downarrow\; 0} + \underbrace{\int_{-C_\varepsilon}}_{\downarrow\; -\pi i}\Bigg)\Bigg] f\, dz = 0$$

$R \to \infty$

$\varepsilon \to 0$

Get $$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

and $$\int_{-\infty}^\infty \frac{\sin t}{t}\, dt = \pi$$

Jordan's lemma yields $P(z), Q(z)$ polys

$$\deg Q \geq \deg P + 1$$ ← instead of $+2$, as before

Q has no zeroes on $\mathbb{R}$.

$$\int_{-\infty}^{\infty} e^{iz} \frac{P(z)}{Q(z)}\, dz = 2\pi i \sum_{a \in \mathcal{O}\!\!\!\mathcal{C}+} \operatorname{Res}_a e^{iz} \cdot \frac{P(z)}{Q(z)}$$

where $\mathcal{O}\!\!\!\mathcal{C}+ = \{\text{zeroes of } Q \text{ in UHP}\}$.

Pf: New twist: estimate $\left| e^{iz} \frac{P(z)}{Q(z)} \right|$ on C_R

Basic poly est $\left| \frac{P(z)}{Q(z)} \right| \leq \frac{B R^{\deg P}}{a R^{\deg Q}}$

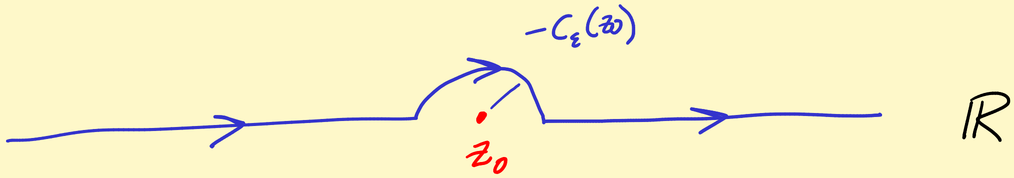
$$\leq \frac{B}{a} \frac{1}{R^{\deg Q - \deg P}}$$ ← ≥ 1

$$\leq \frac{B}{a} \frac{1}{R} \quad \text{when } R > 1$$

Use Jordan's lemma method to estimate

$$|e^{iz}| = e^{-R\sin t} \text{ on } C_R.$$

Another general result.



Fact: If $F(z)$ has a <u>simple</u> pole at z_0, then

$$\lim_{\varepsilon \to 0} \int_{-C_\varepsilon(z_0)} F(z)\, dz = \left[-(2\pi i)\, \mathrm{Res}_{z_0} F \right] \left(\frac{1}{2}\right)$$

<u>Pf</u>:

$$F(z) = \frac{\mathrm{Res}_{z_0} F}{z} + \underbrace{H(z)}_{\text{analytic near } z_0}$$