Lecture 32 The argument principle

HWK due Thurs

Parallel facts about zeroes and poles

Zeroes

$f(z)$ has a zero of order N at z_0

$$f(z) = a_N (z-z_0)^N + \cdots$$

$$= (z-z_0)^N \Big[\underbrace{a_N + a_{N+1}(z-z_0) + \cdots}_{F(z)}\Big]$$

$$f(z) = (z-z_0)^N F(z), \quad F(z_0) = a_N \neq 0$$

Poles

$f(z)$ has a pole of order N at z_0

$$f(z) = \frac{A_N}{(z-z_0)^N} + \cdots + \frac{A_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$= (z-z_0)^{-N} \Big[\underbrace{A_N + A_{N-1}(z-z_0) + \cdots}_{F(z)}\Big]$$

$$f(z) = (z-z_0)^{-N} F(z), \quad F(z_0) = A_N \neq 0$$

Key observation: $n = \pm N \qquad f(z) = (z-z_0)^n F(z), \quad F(z_0) \neq 0.$

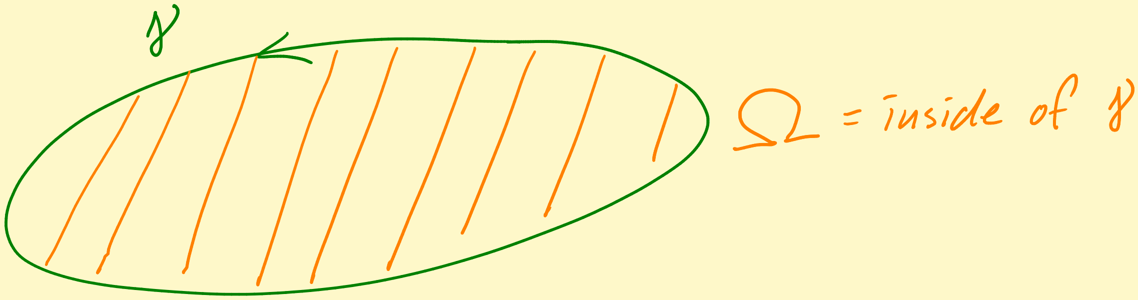
$$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1} F(z) + (z-z_0) F'(z)}{(z-z_0)^n F(z)}$$

$$= \frac{n}{z-z_0} + \underbrace{\frac{F'(z)}{F(z)}}_{\text{analytic near } z_0 !}$$

Wow!
$$\operatorname{Res}_{z_0} \frac{f'}{f} = \begin{cases} \text{Order of zero of } f \text{ at } z_0, & z_0 \text{ a zero} \\ -\text{Order of pole of } f \text{ at } z_0, & z_0 \text{ is a pole} \end{cases}$$

Argument principle

Simple closed curve γ



Case: Zeroes only. f analytic inside and on γ and f has no zeroes on γ. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \begin{pmatrix} \text{\# zeroes of } f \\ \text{inside } \gamma \\ \text{counted with multiplicity} \end{pmatrix}$$

Pf

$$= \frac{1}{2\pi i}\left[2\pi i \sum_{a \in Z_f} \operatorname{Res}_a \frac{f'}{f}\right] = \sum_{k=1}^{N} m_k$$

m_k: orders of zeroes = "multiplicity"

Z_f: finite set $\{a_k\}_{k=1}^N$

because, if it were infinite (in compact set $\overline{\Omega}$), it would have to have a limit pt w_0 in $\overline{\Omega}$.

w_0 Can't be on γ because f not zero there.

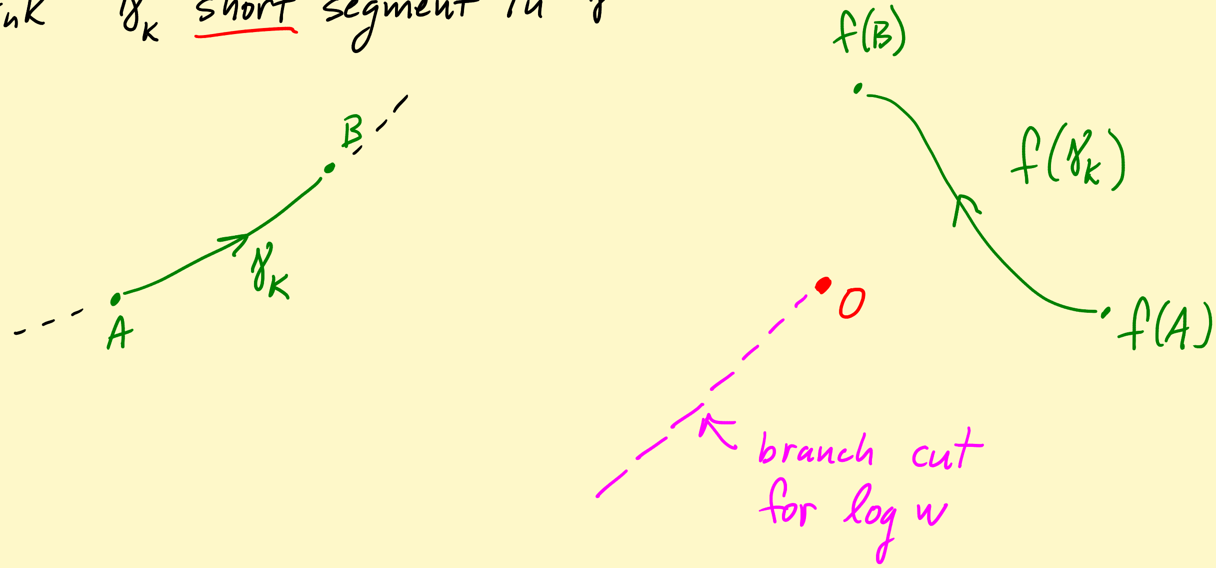
So $w_0 \in \Omega$. Identity thm $\Rightarrow f \equiv 0$. ↯

Case f has zeroes and poles (but none on γ).

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \begin{pmatrix} \text{\# zeroes of } f \\ \text{inside } \gamma \\ \text{counted with mult.} \end{pmatrix} - \begin{pmatrix} \text{\# poles of } f \\ \text{inside } \gamma \\ \text{counted with order} \end{pmatrix}$$

Why "argument" principle.

Think γ_k <u>short</u> segment in γ

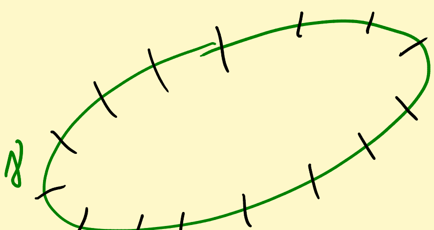


Aha! $$\int_{\gamma_k} \frac{f'}{f}\,dz = \int_{\gamma_k} \frac{d}{dz}\left[\log f\right] dz$$

$$= \log f(z)\Big|_A^B = \log f(B) - \log f(A)$$

$$= \left[\operatorname{Ln}|f(B)| + i\arg f(B)\right] - \left[\operatorname{Ln}|f(A)| + i\arg f(A)\right]$$

$$= \left(\operatorname{Ln}|f(B)| - \operatorname{Ln}|f(A)|\right) + i\,\underbrace{\Delta_{\gamma_k} \arg f}_{\text{does not depend on choice of cut!}}$$



$$\sum \int_{\gamma_k} \frac{f'}{f}\,dz = \underset{\substack{\uparrow \\ \text{real part} \\ \text{sums to zero}}}{0} + i\sum \Delta_{\gamma_k} \arg f$$

Real part: $\left(\mathrm{Ln}|f(z_2)| - \mathrm{Ln}|f(z_1)|\right) + \left(\mathrm{Ln}|f(z_3)| - \mathrm{Ln}|f(z_2)|\right)$

$+ \cdots + \left(\mathrm{Ln}|f(z_N)| - \mathrm{Ln}|f(z_{N-1})|\right)$

Pairwise cancellation, last one cancles first one because γ closed.

$$\int_\gamma \frac{f'}{f}\,dz = i\,\Delta_\gamma \arg f$$

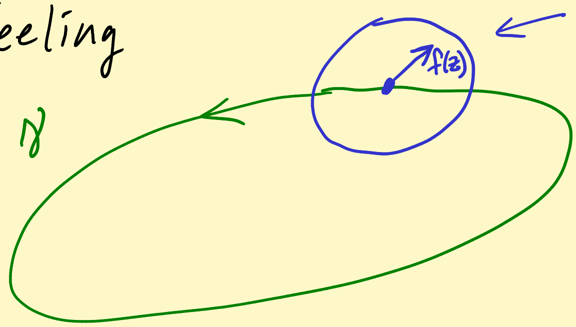
Hmmm. Divide by $2\pi i$. $\dfrac{\Delta_\gamma \arg f}{2\pi}$ = # times $f(z)$ spins around 0 in the counterclockwise direction as z traces out γ.

New improved version

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \left(\begin{matrix}\#\text{ zeroes}\\ f\end{matrix}\right) - \left(\begin{matrix}\#\text{ poles}\\ f\end{matrix}\right) = \frac{\Delta_\gamma \arg f}{2\pi} = \#\text{spins around } 0$$

$f(z)$ = complex number "zeroed" at pt z on γ

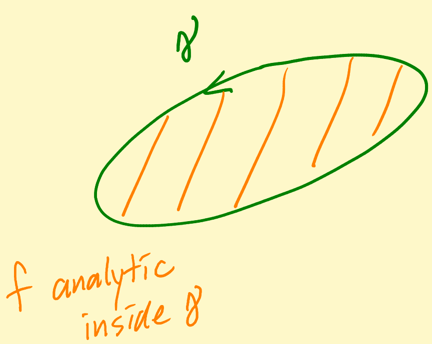
Feeling

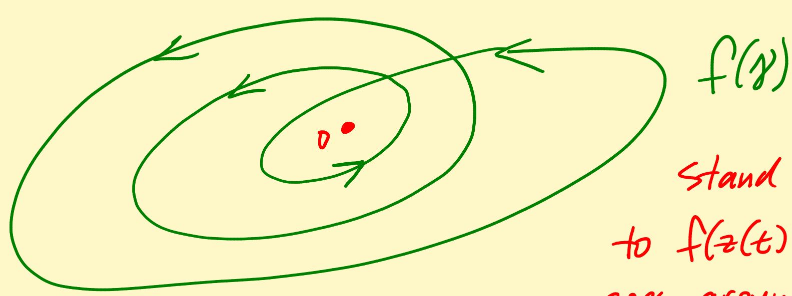


Walk around γ. Count # times $f(z)$ spins around.

Get # zeroes of f inside γ.

(- # poles when poles)





Stand at 0. Point to $f(z(t))$ as $z(t)$ goes around γ.

Spin around 3 times.
f has 3 zeroes inside γ.

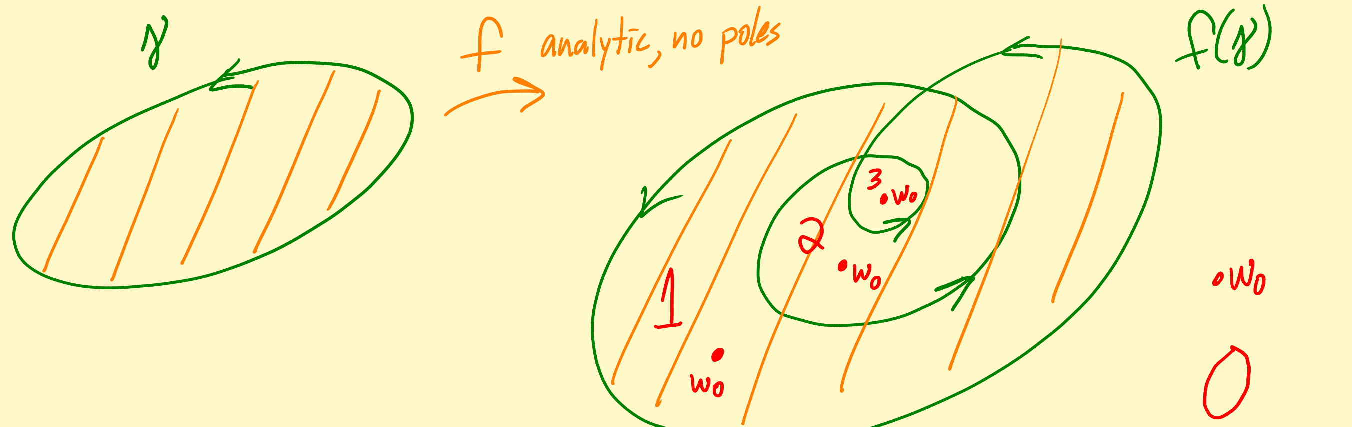
Hmmm:
Could have 3 simple zeroes.
Could have 1 double zero, one simple zero.
Or just <u>one</u> tirple zero.

<u>Nice observation</u> 0 isn't so special. Let $F(z) = f(z) - w_0$

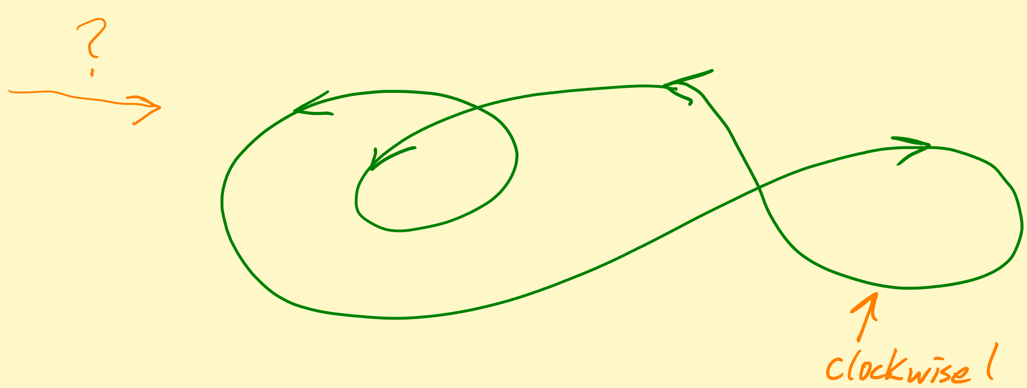
If $F(z_0) = f(z_0) - w_0 = 0$, zero of mult m, then

we say that $f(z_0) = w_0$ with mult m.

$$\frac{F'(z)}{F(z)} = \frac{f'(z)}{f(z) - w_0}$$



Picture?



Must be a <u>pole</u> inside γ

Next time: Rouché's theorem