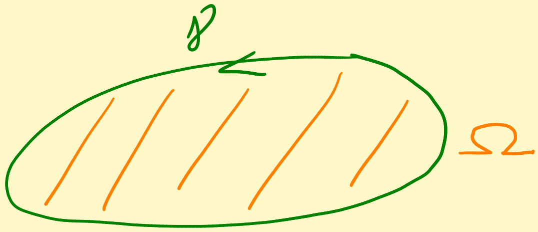
Lecture 33 Rouché's theorem

HWK 10 due TUESDAY, April 22

Rouché's thm



γ simple closed curve, f and h analytic inside and on γ.

If $|h(z)| < |f(z)|$ on γ, then f and $f+h$ have same # of zeroes inside γ counted with multiplicity.

Note: Hypothesis $|h(z)| < |f(z)|$ on γ $\Rightarrow$ f and $f+h$ have no zeroes on γ.

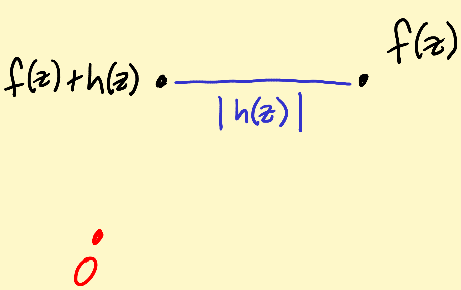
Why: $|h(z)| < |f(z)|$
$\uparrow$ f can't vanish on γ.

Picture see $f+h$ can't vanish either

"Picture proof" $f(z) =$ me
$f(z)+h(z) =$ my dog
$0 =$ telephone pole

$|h(z)| =$ leash length

$\underbrace{|h(z)|}_{\text{leash length}} < \underbrace{|f(z)|}_{\text{distance from me to pole}}$



"See" that dog goes around the pole the same number of times I do (which is the # zeroes of f inside γ).

So $f+h$ has same # by the Argument principle.

EX $\quad \overbrace{\underbrace{z^3 + 9z}_{h} + \underbrace{27}}^{f+h}$ on $D_2(0)$. How many zeroes.

$\uparrow$ big one. $= f(z)$

On $\gamma = C_2(0)$: $\quad |h(z)| = |z^3 + 9z| \le \underbrace{|z|}_{2}^3 + 9\underbrace{|z|}_{2} \leftarrow 2^3 + 9\cdot 2 = 26$

Aha! $\quad |h(z)| \le 26 < 27 = |f(z)|$

So f and $f+h$ have same # zeroes in $D_2(0)$, namely no zeroes.

EX Show that all the zeroes of $z^6 - 5z^2 + 10 = P(z)$ fall in the annulus $\{z : 1 < |z| < 2\}$.

Note: Fund Thm Alg $\Rightarrow$ P has 6 zeroes in $\mathbb{C}$.

On $C_2(0)$: $\quad \overbrace{\underbrace{z^6} - 5z^2 + 10}^{f(z)+h(z)}$ $\qquad$ So $h(z) = -5z^2 + 10$

$\uparrow$ big one $= f(z)$

$$|h(z)| = |-5z^2 + 10| \le 5\cdot 2^2 + 10 = 30 < 64 = |f(z)| = |z^6| = 2^6$$

f has 6 zeroes in $C_2(0)$. So does $f+h = P$.

On $C_1(0)$: $\quad \overbrace{\underbrace{z^6 - 5z^2}_{h(z)} + \underbrace{10}}^{f+h=P}$

$\uparrow$ big one $= f(z)$

no zeroes.

See P has no zeroes inside $D_1(0)$.

[Note: Know hyp $\Rightarrow$ no zeroes on $C_1(0)$ or $C_2(0)$ either.]

So all 6 zeroes are in open annulus.

Why Rouché's is true $\quad |h| < |f|$ on γ

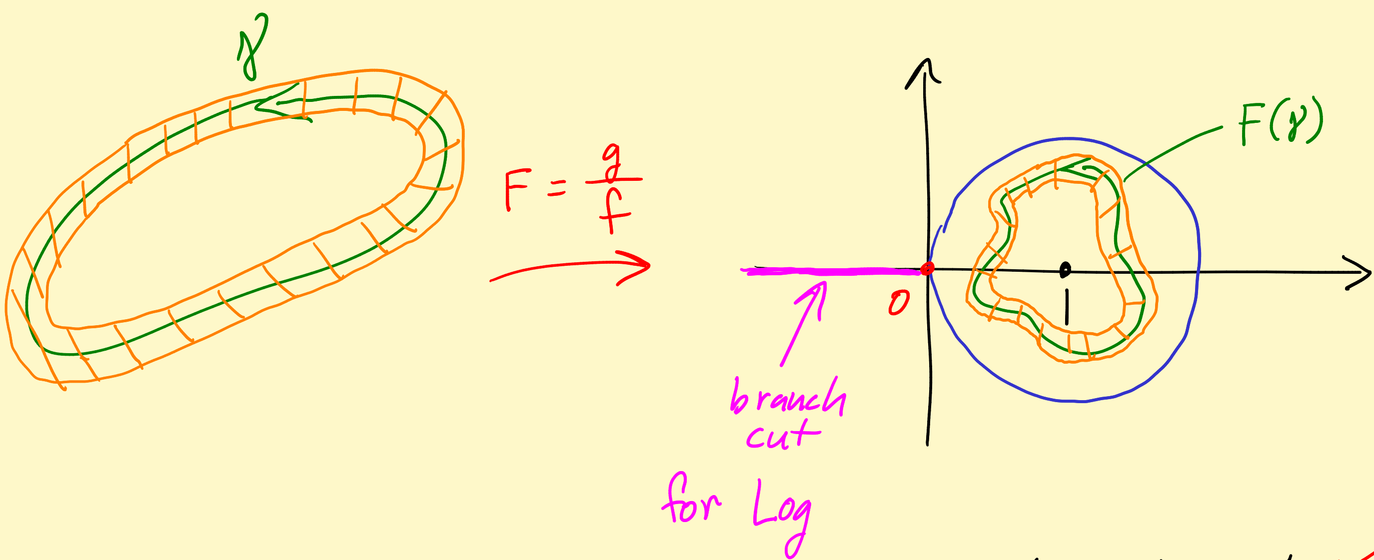
Let $g = f + h$. Want to show f & g have same # zeroes.

$$|h| < |f|$$

$$|f - g| < |f| \longleftarrow$$ some books use f, g instead of f and $f+h$.

Divide by $|f|$: $$\frac{|f-g|}{|f|} < \frac{|f|}{|f|}$$

$$\boxed{\left|1 - \frac{g}{f}\right| < 1 \quad \text{on } \gamma}$$



$F = \frac{g}{f}$ "Logarithmic differentiation" $\quad \frac{F'}{F} = \frac{g'}{g} - \frac{f'}{f} \swarrow$ easy to check

Aha! $\mathrm{Log}\, F(z)$ is analytic in an open set containing γ!

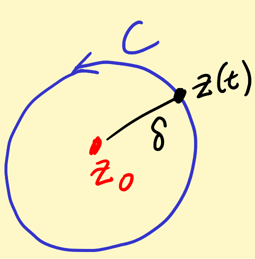
$$\int_\gamma \frac{F'}{F}\,dz = \int_{\underset{\nwarrow \text{closed curve!}}{\gamma}} \frac{d}{dz}\mathrm{Log}\, F(z)\; dz = 0 \quad \text{by Fund Thm Calc.}$$

But $$\frac{1}{2\pi i}\int_\gamma \frac{F'}{F}\,dz = \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{g'}{g}\,dz}_{\#\text{ zeroes } g \text{ inside } \gamma} - \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz}_{\#\text{ zeroes of } f} = 0$$

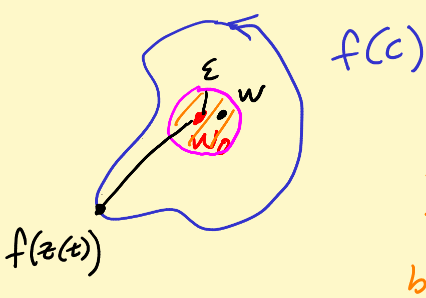
Major consequence for us: Open mapping theorem: Non constant analytic fcns map open sets to open sets.

Why: f not constant. $f(z_0) = w_0$

So $f(z) - w_0$ has an isolated zero at z_0.



Shrink disc down about z_0 so z_0 is only zero of $f(z) - w_0$ in closure



$D_\varepsilon(w_0)$ gets hit by f in $D_\delta(z_0)$

$w \in D_\varepsilon(w_0)$.

Rouché's: $f(z) - w_0$ and $f(z) - w$ have same # zeroes inside C!

$f(z) - w_0$ has one or more.

Same for $f(z) - w$. w gets hit.

Think: Hmmm. $F(z) = f(z) - w_0$

$$F(z) + H(z) = f(z) - w$$

Aha! $H(z) = w_0 - w$

Do I have $|H(z)| < |F(z)|$ on C?

$|w_0 - w| < |f(z) - w_0|$ on C?

Yes! $\varepsilon < \mathrm{dist}(f(z), w_0)$ on C. ✓

Disc $D_\varepsilon(w_0)$ gets hit means f maps open sets to open sets (because z_0 was arbitrary).