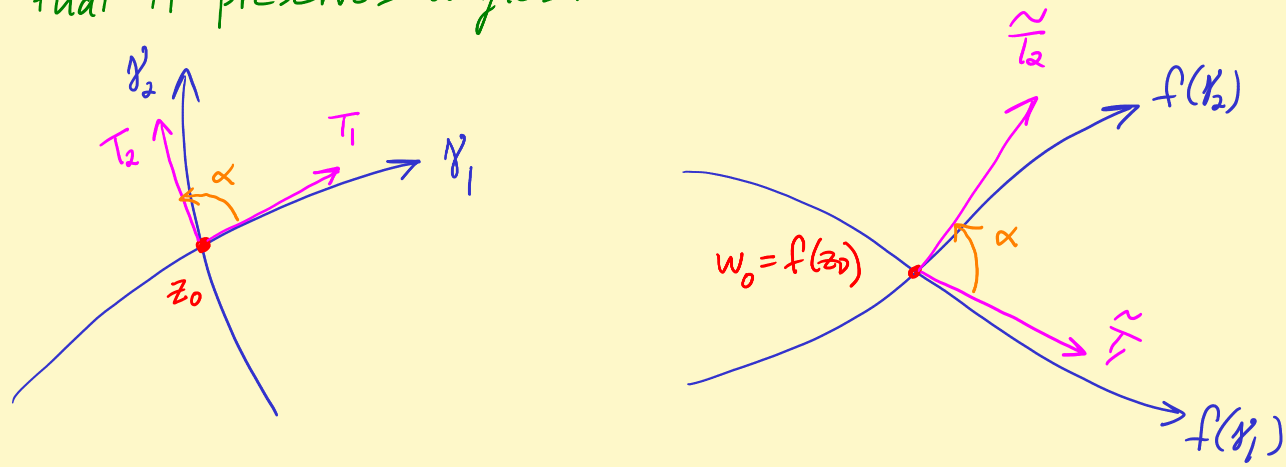
Lecture 34 Conformal mapping

If f is analytic and f' is non-vanishing, then f is <u>conformal</u>, meaning that it preserves angles:



$$\begin{cases} \gamma_j : \; z_j(t) \qquad z_j(0) = z_0 \qquad j = 1, 2 \\ T_j : \; z_j'(0) \end{cases}$$

$\leftarrow$ complex # pointing in direction of the tangent vector

$$f(\gamma_j) : \; f(z_j(t))$$

$$\tilde{T}_j = \frac{d}{dt}\Big[f(z_j(t)) \Big]_{t=0} = f'(z_j(t))\, z_j'(t) \Big|_{t=0}$$

$$= f'(\underbrace{z_j(0)}_{z_0}) \; \underbrace{z_j'(0)}_{T_j}$$

So $$\tilde{T}_j = \underbrace{f'(z_0)}_{Re^{i\theta}} T_j \qquad j = 1, 2$$

Aha! Multiplying by $Re^{i\theta}$ stretches tangent vectors by a factor of $R > 0$, and rotates them by θ in counterclockwise direction. Angle and sense of

α is preserved.

<u>Fun fact</u>: Reverse is true (almost)

$$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

Jacobian matrix $\qquad \mathbb{J} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ $\leftarrow$ matrix mult by $\mathbb{J}$ determines how tangent vectors transform under F

Need to assume u, v are $\mathcal{C}^1$-smooth

and $\det \mathbb{J} \neq 0$ at $z_0 \sim x_0 + iy_0$.

Analytic geometry: linear map is a rotation (preserves angles) if and only if it has form

$$\mathbb{J} = R \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ $\leftarrow$ See C-R equs here!

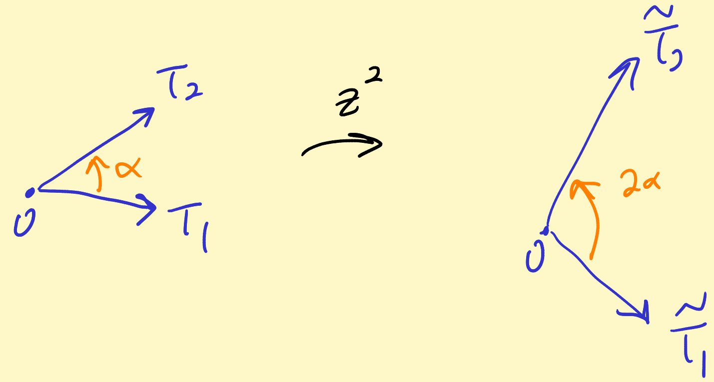
<u>EX</u> $f(z) = z^2$ is <u>not</u> conformal at $z=0$. [$f'(z) = 2z$ is zero at $z=0$.]

It <u>doubles</u> angles at $z=0$.

<u>Why</u>: $z_1(t) = e^{i\theta_1} t \qquad f(z_j(t)) = \left[e^{i\theta_j} t \right]^2 = e^{i2\theta_j} t^2$

$z_2(t) = e^{i\theta_2} t \qquad$ Angle $\alpha = \theta_2 - \theta_1$ gets doubled

$$2\alpha = 2\theta_2 - 2\theta_1 = 2(\theta_2 - \theta_1)$$



opens "like a fan"

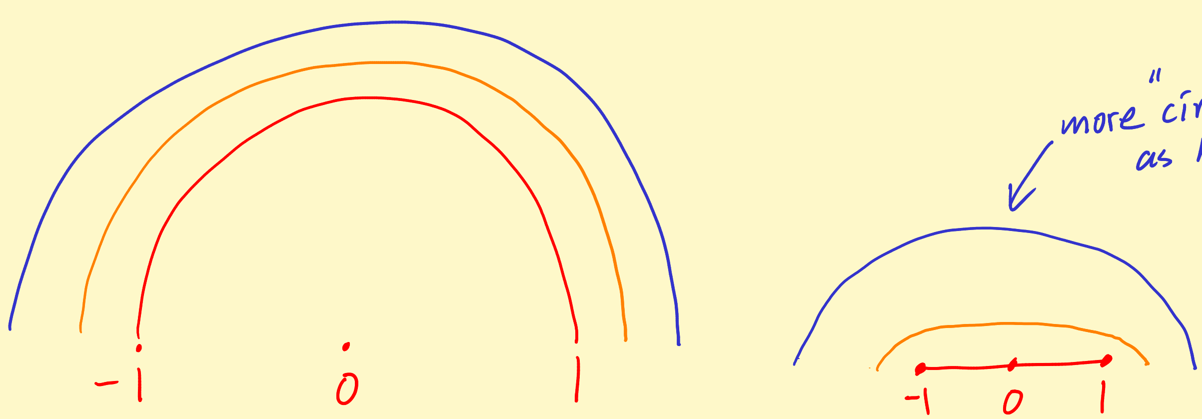
HWK 10 : $z^\alpha = e^{\alpha \operatorname{Log} z}$ ← principal branch

Famous map Jukovsky map $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

On circles: $J(re^{i\theta}) = \frac{1}{2}\left(re^{i\theta} + \frac{1}{re^{i\theta}}\right)$

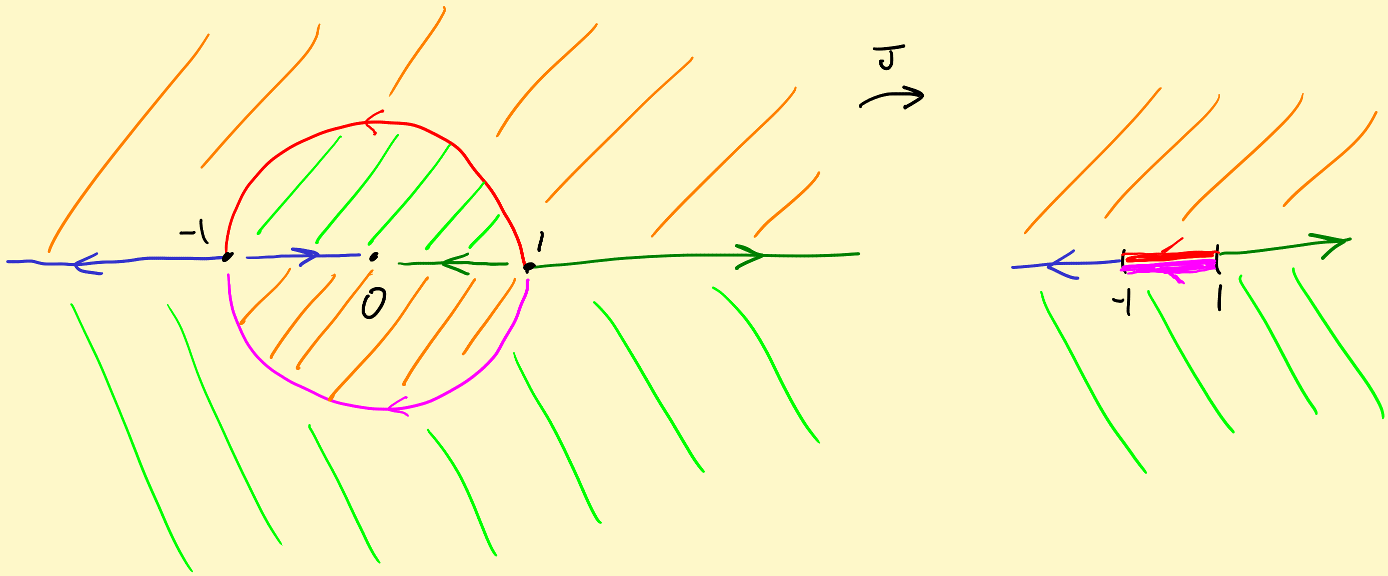
$$= \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right)$$

$$= \underbrace{\left[\frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta\right] + i\left[\frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta\right]}_{\text{ellipse}}$$



more "circular" as $R \to \infty$

See UHP $- \overline{D_1(0)}$: $\xrightarrow[\text{onto}]{1-1}$ UHP

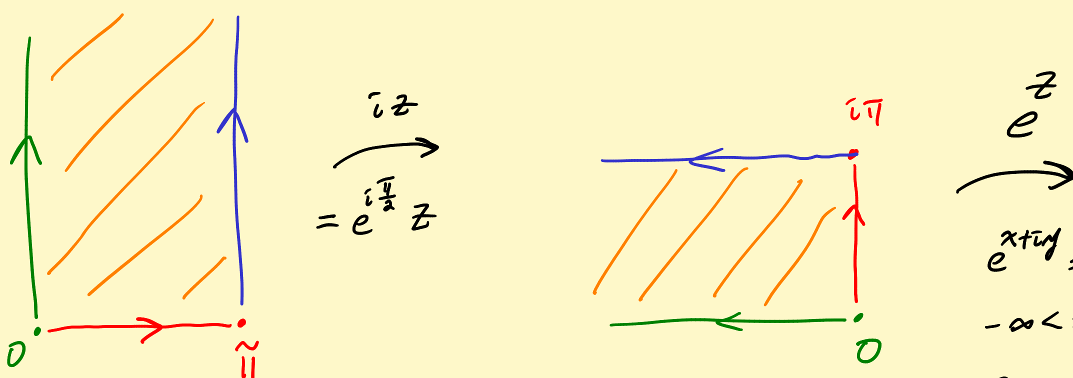


Complex cosine fcn:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$= \frac{1}{2}\left(e^{iz} + \frac{1}{e^{iz}}\right)$$

$$= J\left(e^{iz}\right)$$

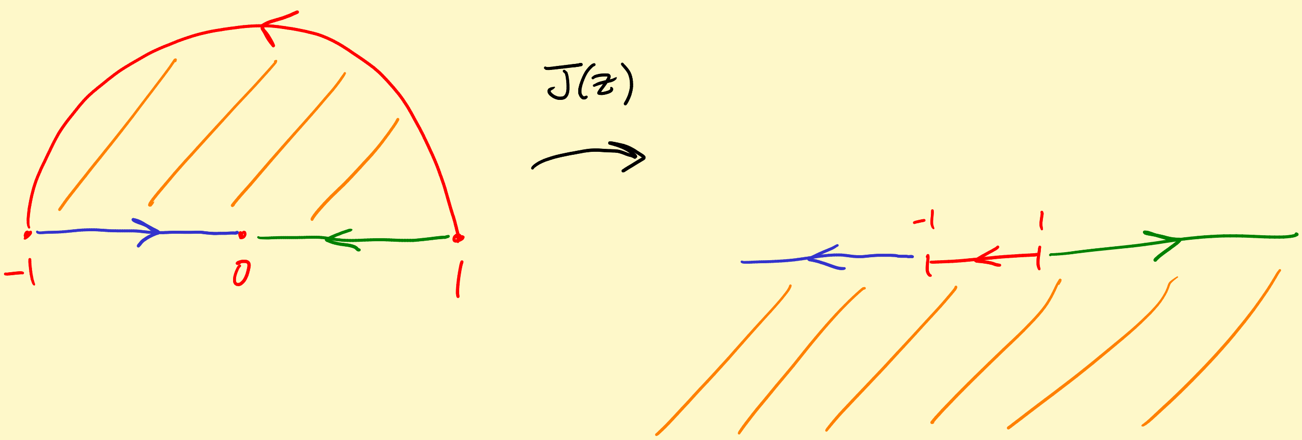


$e^{x+iy} = e^x e^{iy}$

$-\infty < x < 0$

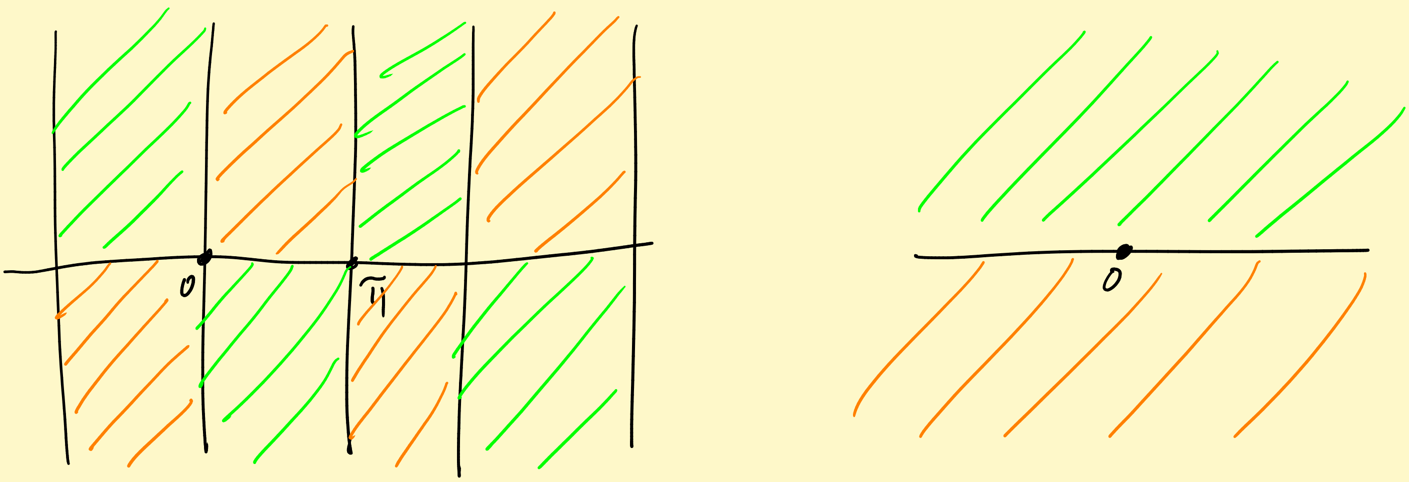
$0 < e^x < 1$

$0 < y < \pi$



Know $\cos z$ on $\mathbb{C}$ now!

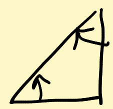
$$\cos(-z) = \cos z$$
$$\cos(z+\pi) = -\cos z$$
$$\cos(z+2\pi) = \cos z$$

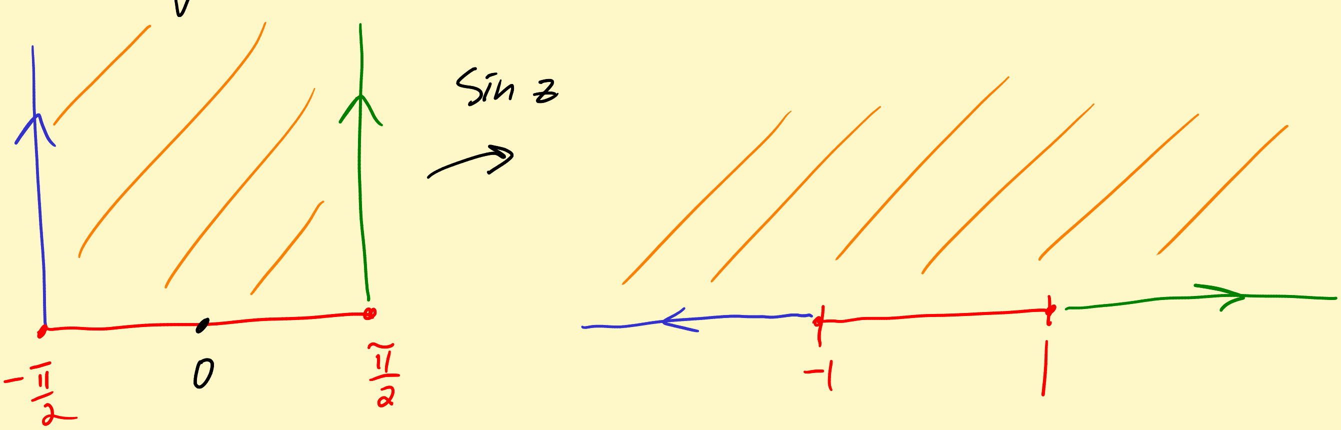


Singularity of a fcn $f(z)$ analytic on $D_\varepsilon(\infty) - \{\infty\} = \{z \in \mathbb{C} : |z| > \frac{1}{\varepsilon}\}$ is defined to be the type of sing of $f(\frac{1}{z})$ at $z = 0$.

See that $\cos z$ has an essential singularity at ∞.

Understanding $\sin z$. $\quad \sin z = \cos(z - \frac{\pi}{2})$





Similar "checkerboard" picture on $\mathbb{C}$.