Lecture 35 Linear fractional transformations (LFT's)

$$L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

[Book call LFTs Möbius transformations. I, and many other books: Möbius LFTs are $\frac{z-a}{1-\bar{a}z}$ ← 1-1, onto analytic $D_1(0) \to D_1(0)$, $a \in D_1(0)$.

Note $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ← Det $= 0$ ⇒ one row is a multiple of the other. Case $c=0, d=0$. $L(z)$ not defined. Otherwise $L(z) \equiv$ const.

LFTs include

1. $z \mapsto rz$ $(r>0)$ "stretch"
2. $z \mapsto e^{i\theta} z$ "rotation"
3. $z \mapsto z+b$ $(b \in \mathbb{C})$ "translation"
4. $z \mapsto \frac{1}{z}$ "inversion"

$a=0, b=1$
$c=1, d=0$

Big fact Basic maps 1-4 generate LFT's

Easy: $c \neq 0$ case $L(z) = \frac{az+b}{cz+d}$, $ad-bc \neq 0$

$$\begin{array}{r|l} & \frac{a}{c} \\ \hline cz+d & az+b \\ & az+\frac{ad}{c} \\ \hline & \left(b-\frac{ad}{c}\right) \end{array}$$

← remainder $\neq 0$

So $$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{\left(b-\frac{ad}{c}\right)}{cz+d}$$

$c = re^{i\theta}$

$b - \frac{ad}{c} = Re^{i\psi}$

$z \mapsto$ stretch by r, rotate by θ, translate by d, invert, stretch by R, rotate by ψ, translate by $\frac{a}{c}$

Case $c=0$: $L(z) = \dfrac{az+b}{d} = \left(\dfrac{a}{d}\right)z + \dfrac{b}{d} = Az+B$

only needs 1–3 to cook it up.

Observation Basic maps 1–3 : $\mathbb{C} \xrightarrow[\text{onto}]{1-1} \mathbb{C}$

$\lim\limits_{z\to\infty} L(z) = \infty$. Write $L(\infty)=\infty$

Then $L: \hat{\mathbb{C}} \xrightarrow[\text{onto}]{1-1} \hat{\mathbb{C}}$

Basic map #4 : $L(z) = \frac{1}{z}$ $L^{-1}(z) = \frac{1}{z}$

$\lim\limits_{z\to 0} \dfrac{1}{z} = \infty$ Write $L(0)=\infty$

$\lim\limits_{z\to\infty} \dfrac{1}{z} = 0$ $L(\infty)=0$

$L: \mathbb{C}-\{0\} \xrightarrow[\text{onto}]{1-1} \mathbb{C}-\{0\}$

$\hat{\mathbb{C}} \xrightarrow[\text{onto}]{1-1} \hat{\mathbb{C}}$

Conclude LFTs map $\hat{\mathbb{C}} \circlearrowleft$ $\underset{\text{onto}}{1-1}$

Case $c=0$. $L(z) = Az+B$. $\infty \longmapsto \infty$

Case $c \neq 0$. $L(z) = \dfrac{az+b}{cz+d}$

$$\lim_{z\to\infty} L(z) = \lim_{z\to\infty} \frac{a+\frac{b}{z}}{c+\frac{d}{z}} = \frac{a}{c}$$

L blows up when denom $= 0$, i.e., when $z = \frac{-d}{c}$

Write $\quad L(\infty) = \frac{a}{c}$

$$L\left(\frac{-d}{c}\right) = \infty$$

Inverse of an LFT is an LFT!

$$w = \frac{az+b}{cz+d}$$

$$czw + dw = az + b$$

$$z(cw - a) = -dw + b$$

$$z = \frac{-dw+b}{cw-a} \cdot \frac{(-1)}{(-1)}$$

$$z = \frac{dw-b}{-cw+a}$$

Got $\quad L^{-1}(w) = \frac{dw-b}{-cw+a}$ $\quad$ Same det $\quad ad - bc \neq 0$ ✓

Hmmm

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\text{Det}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Aha!

$$L \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$L^{-1} \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$$

$\frac{1}{\text{Det}}$ cancels out in L^{-1} (in num and denom)

<u>Cool thing</u> $$L_1(L_2(z)) = \frac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1}$$

$$= \frac{Az + B}{Cz + D}$$

where $$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

<u>Fact</u> Map $L \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbb{A}$

is a "Group homomorphism"

LFT's $\longrightarrow$ 2×2 matrices with $\text{Det} \neq 0$.

<u>Fact</u> LFTs form a group under composition.

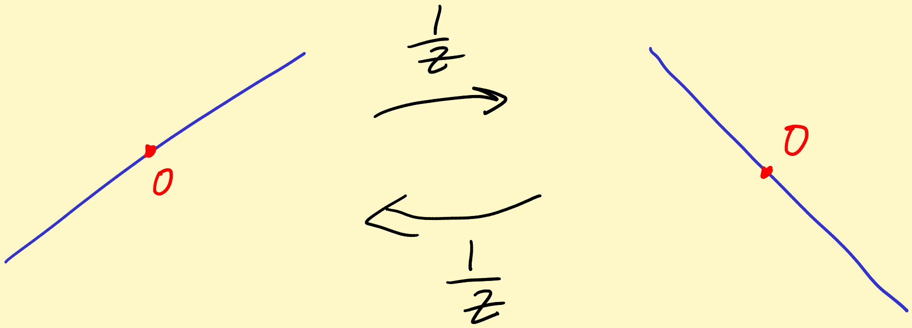
$$id = L(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1} = z \quad \checkmark$$

<u>Geometric facts</u> : LFTs map $\left\{ \begin{matrix} \text{lines} \\ \text{circles} \end{matrix} \right\}$

<u>Pf</u> Basic maps 1–3 : $\{\text{lines}\} \longrightarrow \{\text{lines}\}$

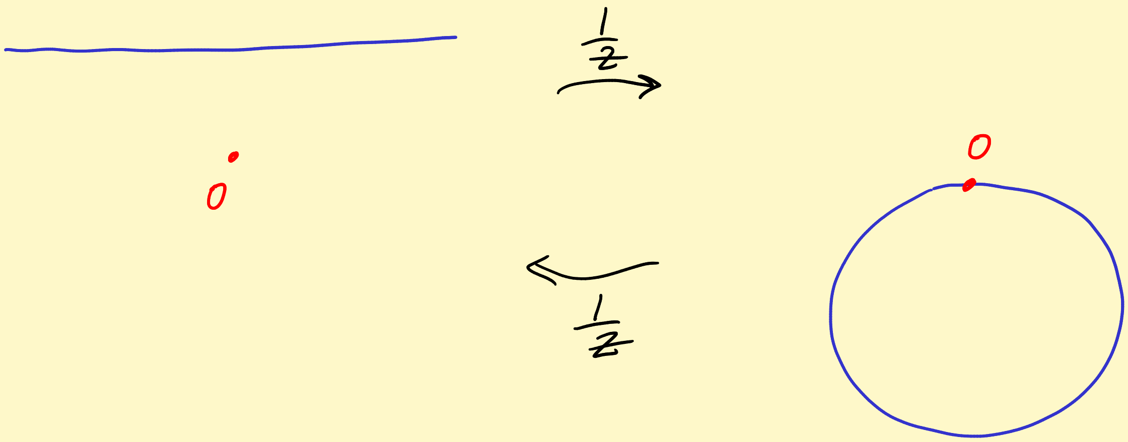
$\{\text{circles}\} \longrightarrow \{\text{circles}\}$

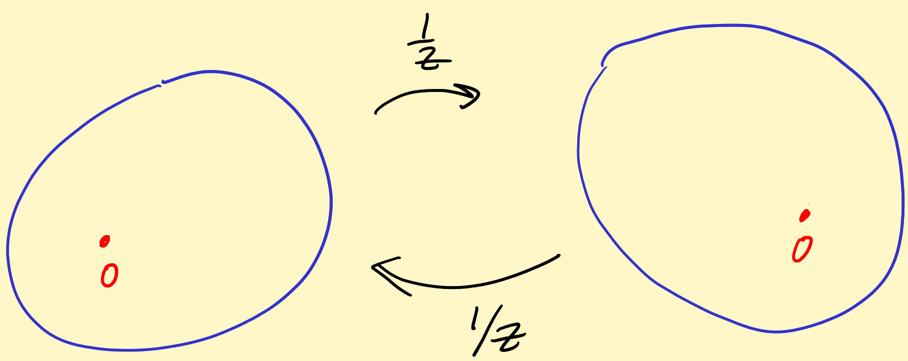
Basic map #4 $\frac{1}{z}$ can mix them up

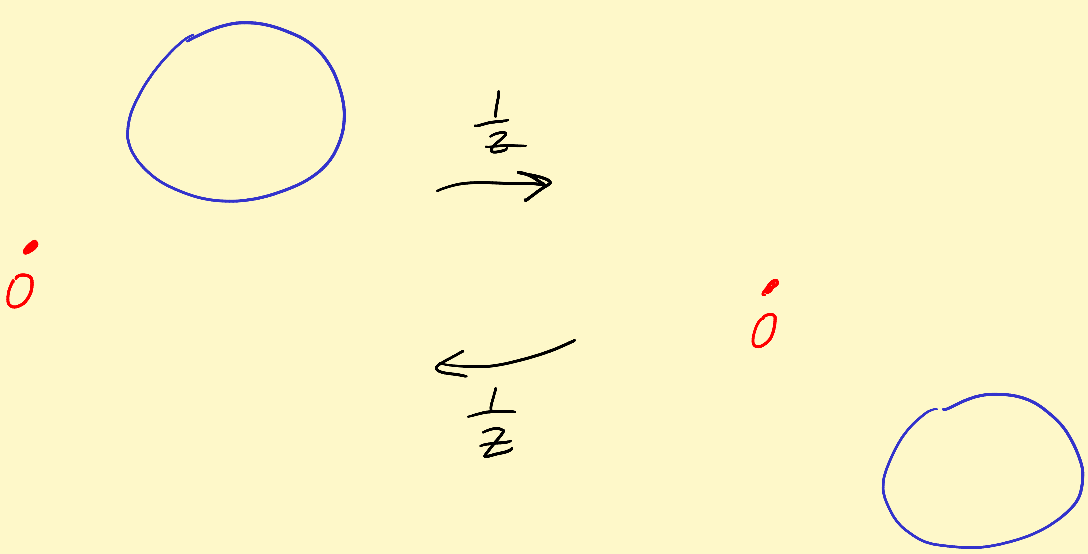


$0 \mapsto \infty$

$\infty \mapsto 0$

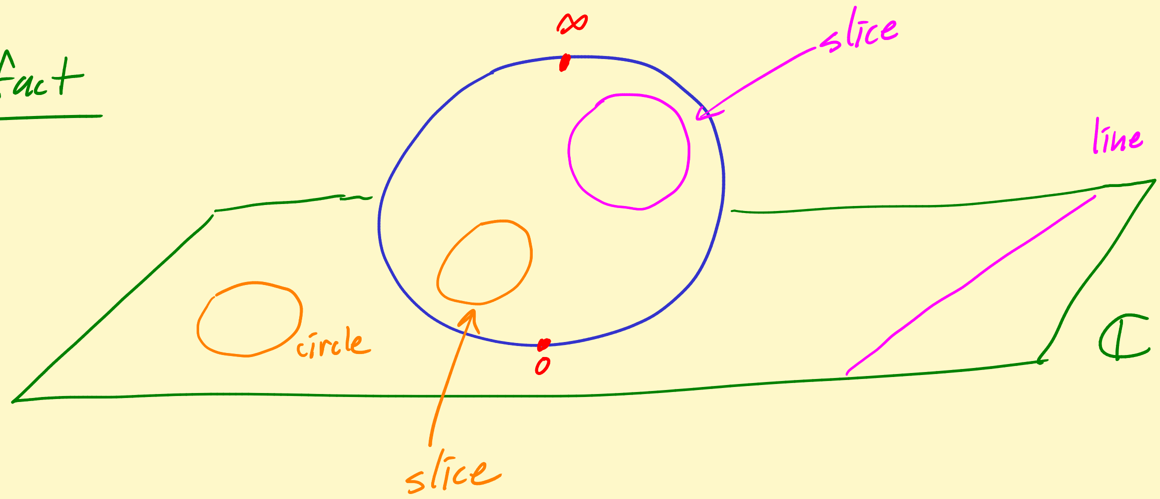






Pf: analytic geometry!

Fascinating fact



$$\left\{\begin{matrix}\text{lines}\\ \text{circles}\end{matrix}\right\} \longleftrightarrow \{\text{slices}\}$$

Big fact

An LFT that fixes 3 pts is the identity.

why: $L(z_j) = z_j \quad j = 1, 2, 3$

$c \neq 0$

$$z_j = \frac{az_j + b}{cz_j + d}$$

$$c z_j^2 + (a-d) z_j - b = 0$$

Hmmm. Quadratic eqn with $\underline{\underline{3}}$ roots?

Must be that $c = 0$, $a - d = 0$, $b = 0$

$a = d$

$ad - bc \neq 0$ ($b = 0$, $c = 0$)

$ad \neq 0$

So $a \neq 0$, $d \neq 0$.

$$L(z) = \frac{az + 0}{0 \cdot z + d} = \left(\frac{a}{d}\right) z = z \quad \checkmark$$

$c = 0$. Easier. $L(z) = Az + B$

Cor $L_1 = L_2$ at 3 pts $\Rightarrow$ $L_1 = L_2$

Fact 3 pts in $\mathbb{C}$ determine a circle.

2 pts (plus the pt at infinity) determine a line.