Lecture 37 Applications of LFT's

1. LFT's $\left\{ \begin{matrix} \text{lines} \\ \text{circles} \end{matrix} \right\}$

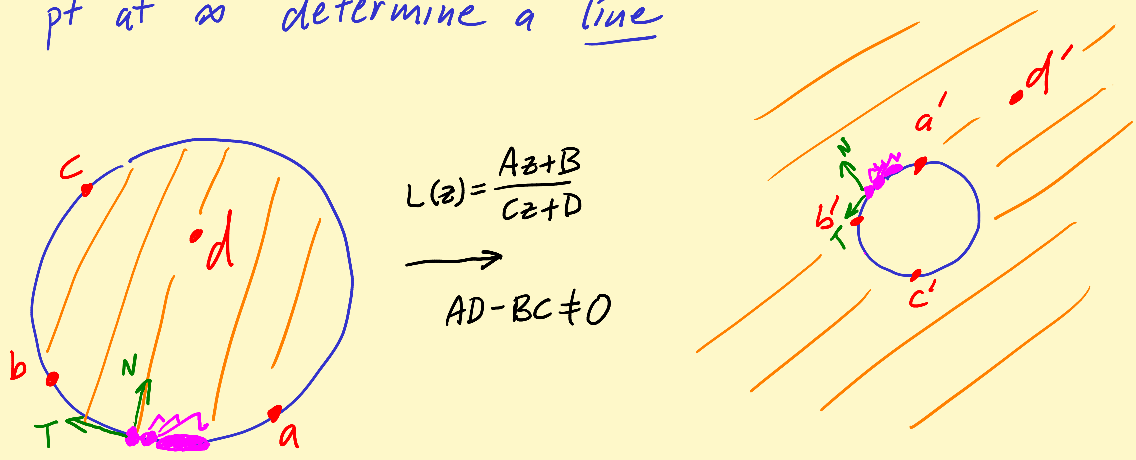
2. LFT's that agree at 3 pts are the same

3. LFT's : $\hat{\mathbb{C}} \xrightarrow[\text{onto}]{1-1} \hat{\mathbb{C}}$ & L and L^{-1} are continuos on $\hat{\mathbb{C}}$

4. 3 pts in $\mathbb{C}$ determine a <u>circle</u>

   2 pts + pt at ∞ determine a <u>line</u>

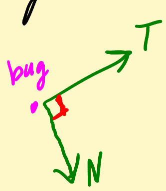
<u>Pictures</u>



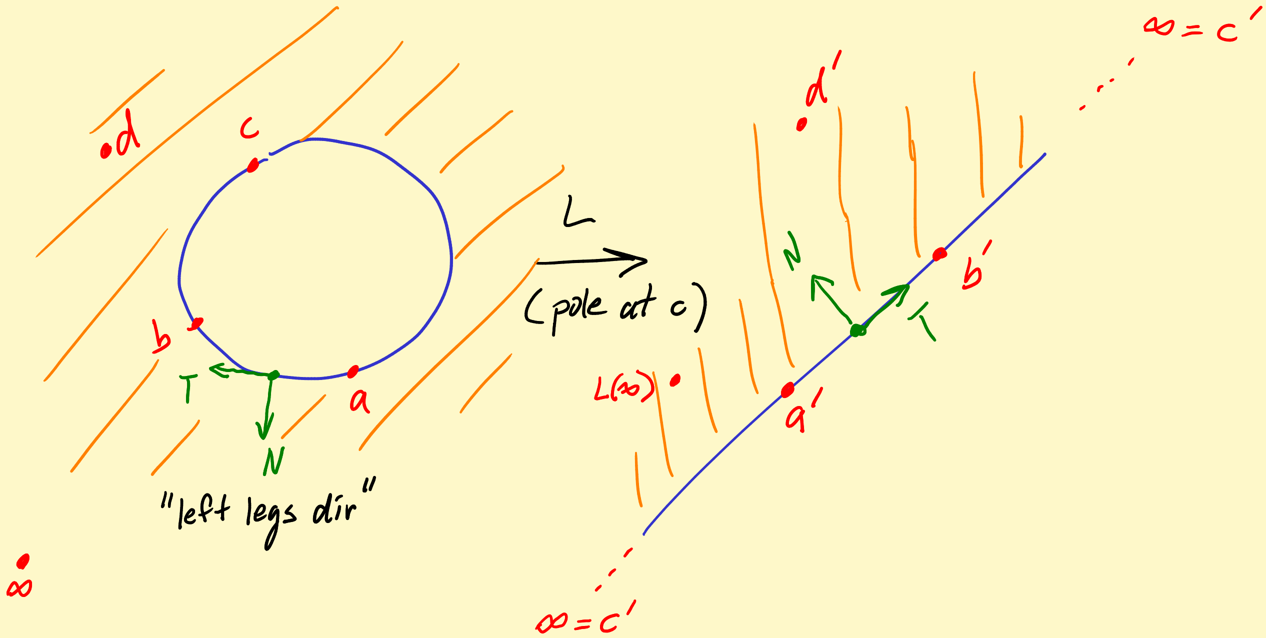
Ques: Where does inside of circle go?

<u>One pt test</u> Pick any pt in a connected component of $\mathbb{C} - \{\text{circle}\}$. It maps 1-1 onto connected component of the image determined by the image of the pt.

<u>"Bug" test</u> As bug crawls from a to b, image bug crawls from a' to b' and legs determine which side goes where by conformality



Another possible picture



L maps circle to <u>line</u>. Can use <u>two</u> pts, a & b to determine line. Bug test or one pt test determine where connected components go. [Need to think about one thing.

$$L(\infty) = \lim_{z \to \infty} \frac{Az+B}{Cz+D} = \frac{A}{C} \quad]$$

missing pt in image picture

How to cook up LFTs

Easy one: $\quad \dfrac{z-a}{z-b} \qquad \begin{array}{l} a \mapsto 0 \\ b \mapsto \infty \end{array}$

Hmmm. What if I wanted $c \mapsto 1$ too.

$$L(z) = \frac{c-b}{c-a} \cdot \frac{z-a}{z-b} \qquad L: \begin{array}{l} a \mapsto 0 \\ b \mapsto \infty \\ c \mapsto 1 \end{array}$$

Problem Find an LFT mapping $\begin{array}{l} z_1 \mapsto w_1 \\ z_2 \mapsto w_2 \\ z_3 \mapsto w_3 \end{array}$ (pts in $\mathbb{C}$)

[Maps $\{z_1, z_2, z_3\}$ circle $\longrightarrow$ $\{w_1, w_2, w_3\}$ circle.]

Sol^n: $L_1(z) = \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{z - z_1}{z - z_2}$ $\quad$ $z_1 \mapsto 0$, $z_2 \mapsto \infty$, $z_3 \mapsto 1$

$L_2(w) = \dfrac{w_3 - w_2}{w_3 - w_1} \cdot \dfrac{w - w_1}{w - w_2}$ $\quad$ $w_1 \mapsto 0$, $w_2 \mapsto \infty$, $w_3 \mapsto 1$

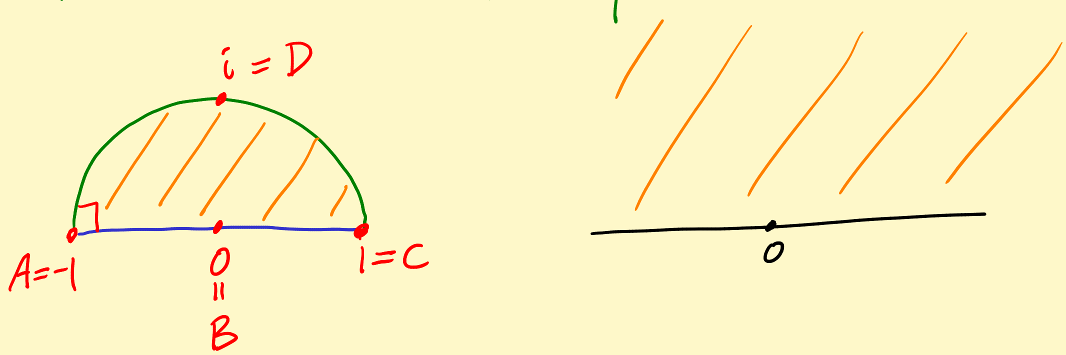
Ans: $L_2^{-1} \circ L_1$

Computing $L_1 \sim \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$L_2 \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ $\qquad$ $L_2^{-1} \sim \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$

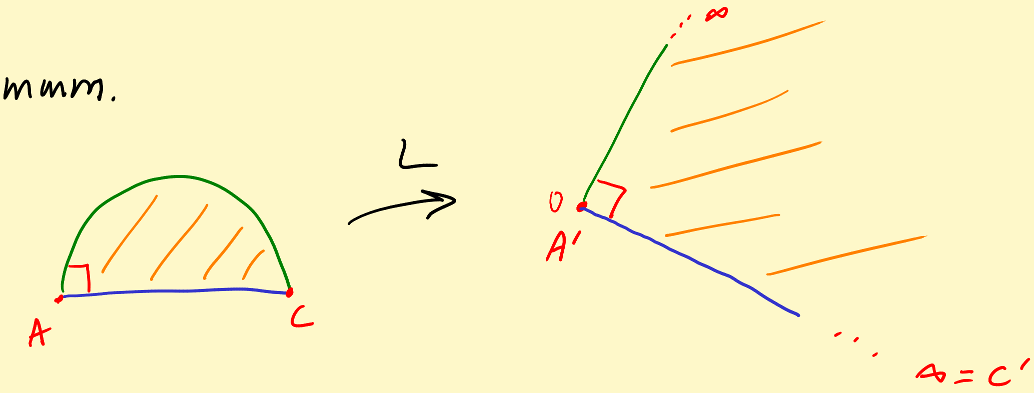
$$L_2^{-1} \circ L_1 \sim \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

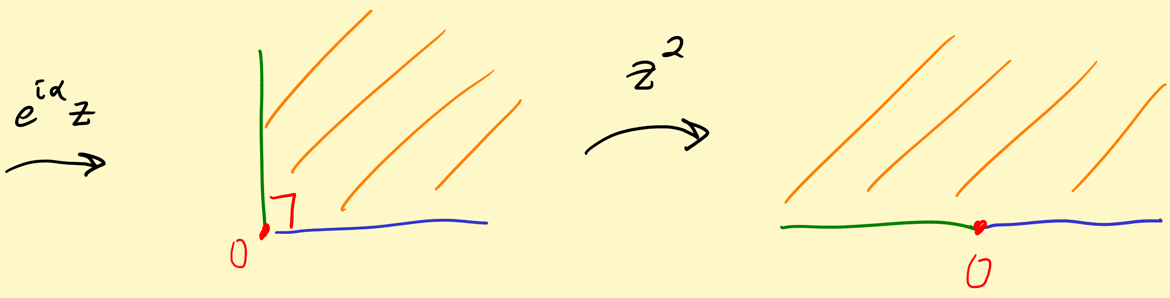
Prob Find a conformal map



Idea An LFT that maps C to ∞ would map blue line to a line and green circle both to lines.

Hmmm.





<u>Details</u>

$$L(z) = \frac{z-(-1)}{z-1} = \frac{z+1}{z-1}$$

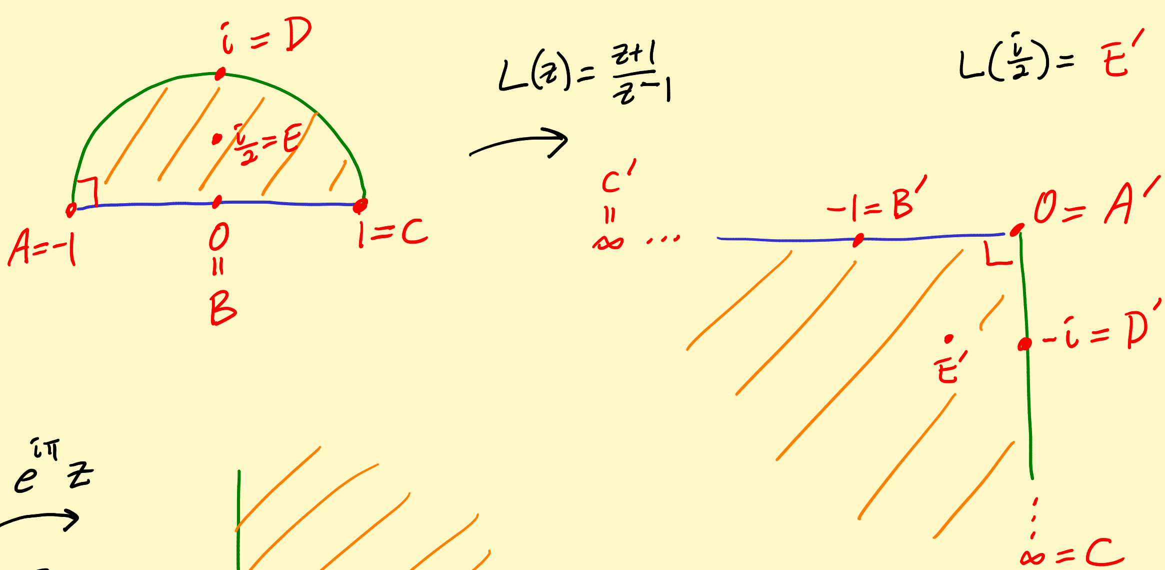
$L(1) = \infty = C'$

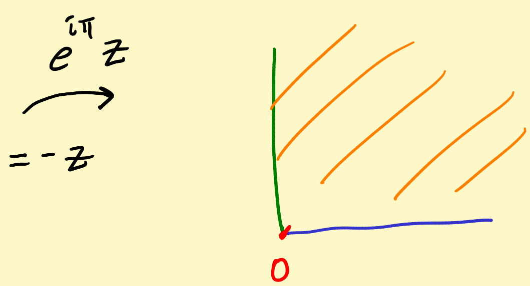
$L(-1) = 0 = A'$

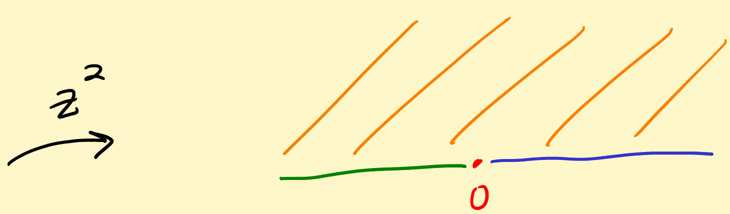
$L(i) = \frac{i+1}{i-1} = -i = D'$

$L(0) = \frac{0+1}{0-1} = -1 = B'$

$L(\frac{i}{2}) = E'$ or use bug test



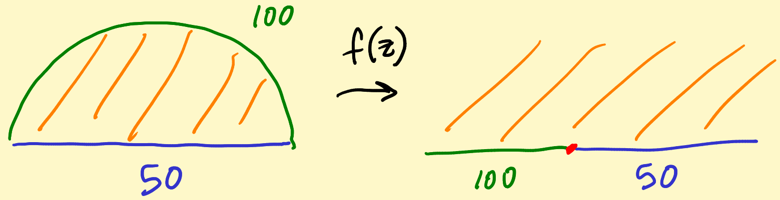




Composition:

$$f(z) = \left[-\left(\frac{z+1}{z-1}\right)\right]^2 = \left(\frac{z+1}{z-1}\right)^2$$

Heat problem



Sol^n

$$T(z) = \frac{50}{\pi}\,\mathrm{Arg}\, z + 50$$

$$T(f(z))$$

Big picture