Lecture 39 Schwarz lemma

Schwarz lemma $f: D_1(0) \circlearrowleft$ analytic and $f(0)=0$.

[$|f(z)|<1$ when $|z|<1$.]

Then A) $|f(z)| \le |z|$ for $z \in D_1(0)$, and

B) $|f'(0)| \le 1$.

Furthermore, if equality holds in (A) for some $z \neq 0$, or if equality holds in (B), then $f(z) \equiv \lambda z$ for some unimodular constant $\lambda = e^{i\theta}$. [i.e., $f(z)$ is just a rotation.]

Pf Define

$$F(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

Note: $DQ = \dfrac{f(z)-f(0)}{z-0} = \dfrac{f(z)}{z} \longrightarrow f'(0)$ as $z \to 0$.

See $F(z)$ is continuous at $z=0$.

Aha! $z=0$ is a removable sing for F.

Idea: Max princ on $D_r(0)$ and $r \nearrow 1$.

Let $z_0 \in D_1(0)$ and r with $|z_0| < r < 1$.

Then $|F(z_0)| \le \underset{D_r(0)}{\mathrm{Max}} |F| = \underset{z \in C_r(z)}{\mathrm{Max}} \left| \dfrac{f(z)}{z} \right|$ $\leftarrow |f(z)| \le 1$, $\leftarrow |z| = r$ on $C_r(0)$

Get $|F(z_0)| \le \frac{1}{r}$.

Aha! Can let $r \nearrow 1$. Then $\frac{1}{r} \searrow 1$ and we see

that $|F(z_0)| \le 1$.

Since z_0 was arbitrary, $|F| \le 1$ on $D_1(0)$.

A ✓
B ✓

$z \ne 0$. But $f(0)=0$.
so true at $z=0$ too.

Second part of proof:

If equality holds in A) for some $z_0 \ne 0$, or if equality holds in B) (at $z_0 = 0$), then $|F(z_0)| = 1$ at $z_0 \in D_1(0)$. But we showed $|F(z)| \le 1$ on $D_1(0)$.

F analytic and $|F|$ max at $z_0 \Rightarrow F \equiv$ const λ.

$|F(z_0)| = 1 \Rightarrow |\lambda| = 1$. ✓

Big consequence of Schwarz The "automorphism group" of the unit disc $Aut(D_1(0)) = \{ f : D_1(0) \circlearrowleft \text{ analytic}, \text{ 1-1 onto}\}$

is given by $\left\{ \lambda \frac{z-a}{1-\bar{a}z} : |\lambda| = 1,\ a \in D_1(0) \right\}$

= the Möbius transformations.

why: Step 1 $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$ $a \in D_1(0)$.

Claim $\varphi_a : D_1(0) \circlearrowleft \overset{1\text{-}1}{\text{onto}}$. Pick a pt e^{it} on $C_1(0)$.

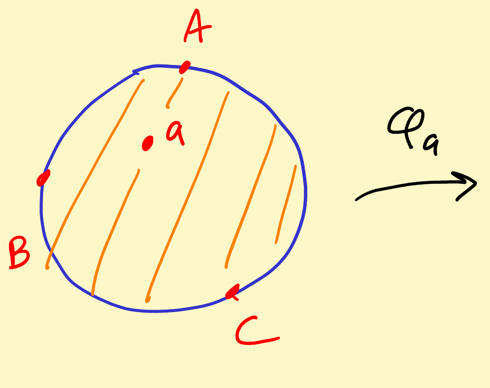
$$\varphi_a(e^{it}) = \frac{e^{it} - a}{1 - \bar{a}e^{it}} \cdot \frac{e^{-it}}{e^{-it}} = \frac{e^{-it}(e^{it} - a)}{e^{-it} - \bar{a}}$$

conjugates

$$= e^{-it} \cdot \frac{w}{\bar{w}}$$

both unimodular

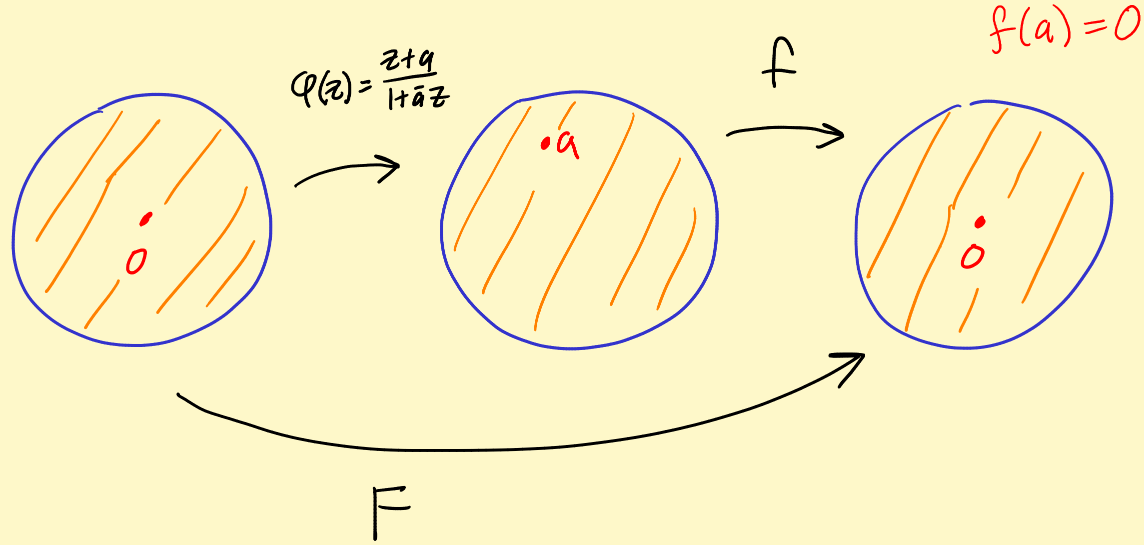
See $\varphi_a : C_1(0) \circlearrowleft$



LFT's: One pt test. $D_1(0) \xrightarrow[\text{onto}]{1\text{-}1} D_1(0)$

Note: $\varphi_a(z) = \frac{z-a}{1-\bar{a}z} = \frac{z-a}{-\bar{a}z+1}$ so $\varphi_a^{-1}(z) = \frac{z+a}{1+\bar{a}z}$

Step 2 Suppose $f \in Aut(D_1(0))$



$F : D_1(0) \circlearrowleft \overset{1\text{-}1}{\text{onto}}$, $F(0) = 0$. Schwarz! B) $|F'(0)| \le 1$

But, F^{-1} also satisfies same conditions. So $|(F^{-1})'(0)| \le 1$.

Note: $(F^{-1})'(0) = \dfrac{1}{F'(F^{-1}(0))} = \dfrac{1}{F'(0)}$. So $|F'(0)| \ge 1$ too.

Conclude $|F'(0)| = 1$. Schwarz, part 2 $\Rightarrow$

$$F(z) = \lambda z, \quad |\lambda| = 1.$$

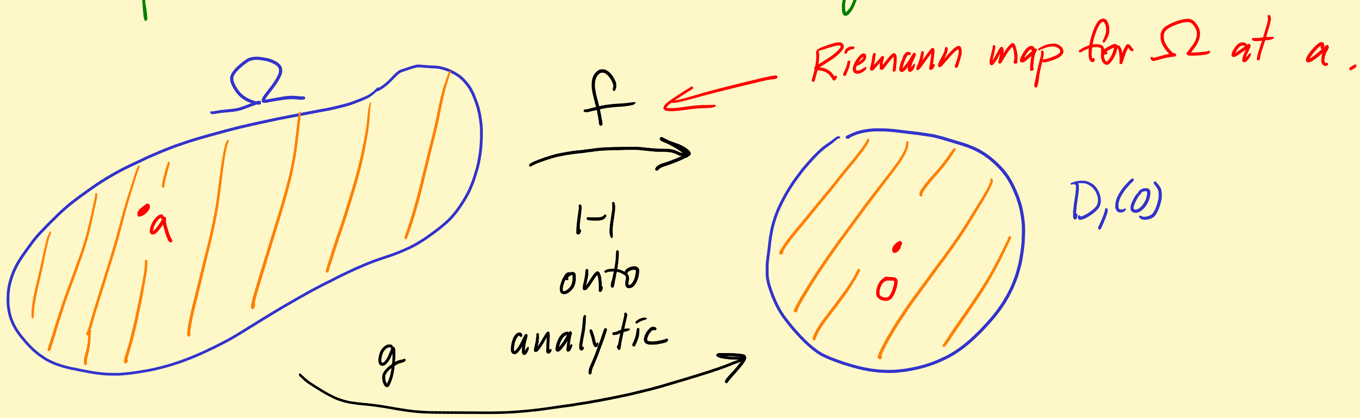
Done: $f(\underbrace{\varphi(z)}_{w}) = \lambda z$

$w = \varphi(z)$

$z = \varphi^{-1}(w) = \dfrac{w-a}{1-\bar{a}w}$

$$f(w) = \lambda \frac{w-a}{1-\bar{a}w} \quad \checkmark$$

Important fact: Schwarz lemma is a key ingredient in the proof of the Riemann mapping thm.



Suppose $g: \Omega \to D_1(0)$, analytic with $g(a) = 0$.

Schwarz $\Rightarrow$ $|g'(a)| \le |f'(a)|$

Why: $g \circ f^{-1}: D_1(0) \to D_1(0)$ fixing 0. $F(z) = g(f^{-1}(z))$.

Schwarz (B) $\Rightarrow$ $|F'(0)| \le 1$. $\Rightarrow$ $|g'(a)| \le |f'(a)|$.

Strategy for cooking up Riemann map: Find analytic fcn

$g : \Omega \to D_1(0)$ analytic, <u>maximizing</u> $|g'(a)|$.

It works!

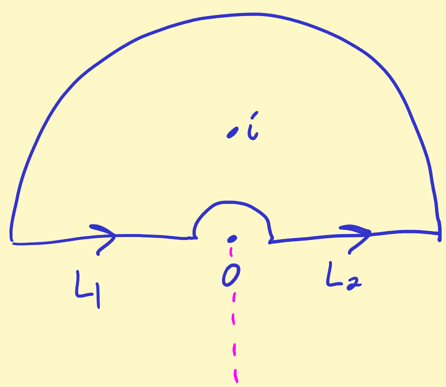
HWK Hint $\qquad \dfrac{z^\alpha}{(z^2+1)^2} \qquad z^\alpha = e^{\alpha \log_0 z}$

$$= \frac{z^\alpha}{\left[(z-i)(z+i)\right]^2} = \frac{1}{(z-i)^2}\Bigg[\underbrace{\frac{z^\alpha}{(z+i)^2}}_{H(z) = A_0 + A_1(z-i) + \cdots}\Bigg]$$

$$= \frac{A_0}{(z-i)^2} + \frac{A_1}{z-i} + \text{power series}$$

Res!

$$\operatorname{Res}_i \frac{z^\alpha}{(z^2+1)^2} = A_1 = \frac{H'(i)}{1!}$$



or "PacMan" contour