Review 1

Final exam Thurs, May 8, 8-10 am SC 239

Find practice prob and an old final on home page.

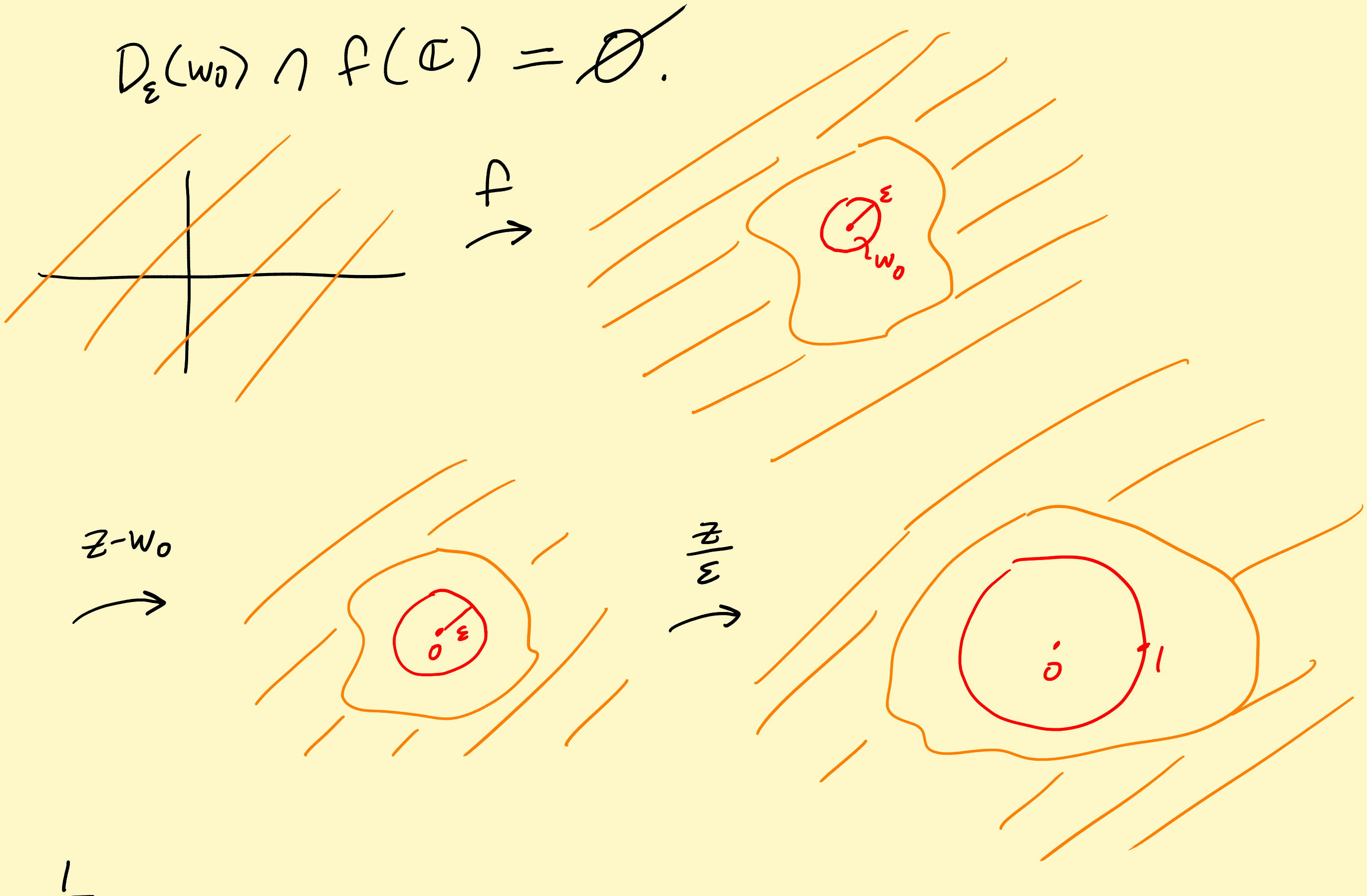
No class on Friday. [2 sheets regular sized paper handwritten, both sides]

Casorati–Weierstraß thm An entire fcn that misses an open set must be constant.

Suppose f analytic on $\mathbb{C}$.

Pf $f(\mathbb{C})$ misses open set $\mathcal{O}$, then $\exists$ $D_\varepsilon(w_0)$ such that

$$D_\varepsilon(w_0) \cap f(\mathbb{C}) = \emptyset.$$



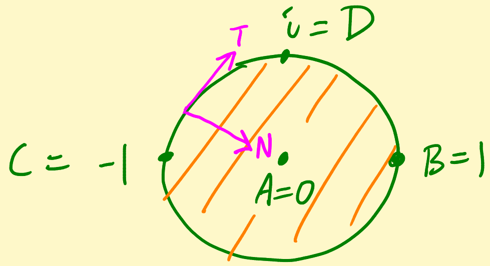
Aha! $\dfrac{\varepsilon}{f(z)-w_0}$ is a bndd entire fcn!

Never zero. Liouville's $\Rightarrow$

$$\frac{\varepsilon}{f(z)-w_0} = c \neq 0.$$

Solve for f. See $f \equiv \text{const}$. ✓

\#7 old exam: $T_1(z) = \dfrac{z}{1-z} \xrightarrow{?}$ on unit disc.

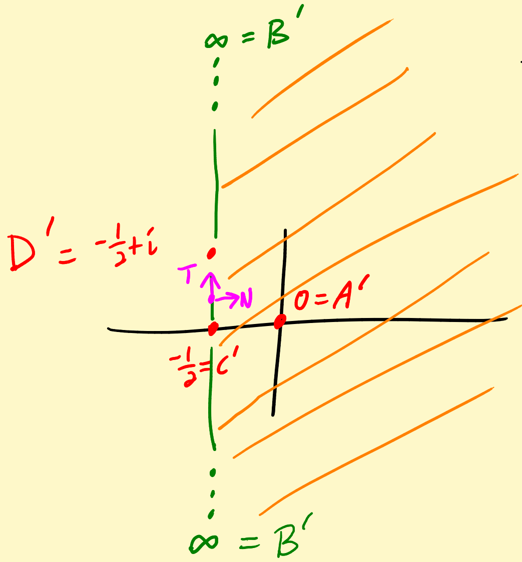


$T_1(0) = 0 = A'$

$T_1(1) = \infty = B'$ ← Aha! $C_1(0)$ maps to a line!

$T_1(-1) = \frac{-1}{2} = C'$

$T_1(i) = \dfrac{i}{1-i} \cdot \dfrac{1+i}{1+i} = \dfrac{-1+i}{2} = D'$



Bug test

<u>Important tool.</u> $\operatorname{Res}_a \dfrac{f}{g} = \dfrac{f(a)}{g'(a)}$ when f, g analytic near a and a is a <u>simple</u> zero of g: $g(a) = 0$, $g'(a) \neq 0$.

What if g has a double zero? $g(a), g'(a) = 0$, $g''(a) \neq 0$.

Hmmm. $\dfrac{f(z)}{g(z)}$ would have a <u>pole</u> of order 2 at a.

$$\frac{f(z)}{g(z)} = \frac{B_2}{(z-a)^2} + \frac{B_1}{z-a} + \text{power series}$$

Res

Books : $(z-a)^2 \dfrac{f(z)}{g(z)} = B_2 + B_1(z-a) + \cdots$ analytic

Hate!: $\dfrac{1}{n!} \lim_{z\to a} \dfrac{d^n}{dz^n}\left[(z-a)^{n+1}\dfrac{f(z)}{g(z)}\right]$

$n=1$ here to get B_1

Better way

$$g(z) = \underset{\overset{\shortparallel}{g(a)=0}}{a_0} + \underset{\overset{\shortparallel}{\frac{g'(a)}{1!}=0}}{a_1}(z-a) + \underset{\overset{\shortparallel}{\frac{g''(a)}{2!}}}{a_2}(z-a)^2 + \cdots$$

$$= (z-a)^2\, G(z) \quad \text{where} \quad G(z) = a_2 + \underset{\overset{\shortparallel}{\frac{g'''(a)}{3!}}}{a_3}(z-a) + \cdots$$

$$\frac{f(z)}{g(z)} = \frac{1}{(z-a)^2}\left[\underbrace{\frac{f(z)}{G(z)}}\right]$$

$$\left[\begin{aligned} G(a) &= a_2 = \frac{g''(a)}{2!} \neq 0 \\ \frac{G'(a)}{1!} &= a_3 = \frac{g'''(a)}{3!} \end{aligned}\right.$$

$H(z) = A_0 + A_1(z-a) + \cdots$

analytic near a

$$= \frac{A_0}{(z-a)^2} + \frac{A_1}{z-a} + \text{power series}$$

$\leftarrow \mathrm{Res}_a$

$$\mathrm{Res}_a \frac{f}{g} = A_1 = \frac{H'(a)}{1!} = \frac{f'(a)G(a) - G'(a)f(a)}{G(a)^2} = \frac{f'(a)\frac{g''(a)}{2!} - \frac{g'''(a)}{3!}f(a)}{\left[\frac{g''(a)}{2!}\right]^2}$$

EX : $\dfrac{e^{2z}}{1-\cos z} = \dfrac{e^{2z}}{\frac{z^2}{2!} - \frac{z^4}{4!} + \cdots}$

$$= \frac{1}{z^2}\left[\underbrace{\frac{e^z}{\frac{1}{2}-\frac{z^2}{4!}+\cdots}}_{\underbrace{A_0 + A_1 z + \cdots}_{H(z)}}\right] = \frac{A_0}{z^2} + \frac{A_1}{z} + \text{power series}$$

$$\text{Res}_0 = A_1 = \frac{H'(0)}{1!}$$

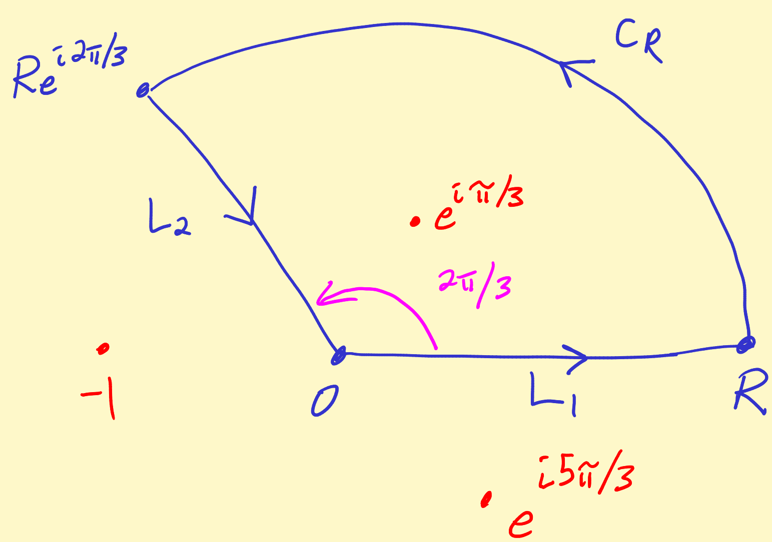
<u>EX</u>

$$\frac{e^z}{(z^2+4)^3} = \frac{e^z}{\left[(z-2i)(z+2i)\right]^3}$$

$$= \frac{1}{(z-2i)^3}\left[\underbrace{\frac{e^z}{(z+2i)^3}}_{H(z) = A_0 + A_1(z-2i) + \cdots}\right]$$

$$= \frac{A_0}{(z-2i)^3} + \frac{A_1}{(z-2i)^2} + \frac{A_2}{(z-2i)} + \cdots \text{ p.s.}$$

$$\text{Res}_{2i} = A_2 = \frac{H''(2i)}{2!}$$



$$\int_0^\infty \frac{dx}{x^3+1}$$

$$f(z) = \frac{1}{z^3+1}$$

Like $t=0 \nearrow R$. $\quad -L_2: z(t) = t\, e^{i2\pi/3}$, $0 < t < R$

$$\int_{L_2} f(z)\,dz = -\int_{-L_2} f(z)\,dz = -\int_0^R \frac{1}{\underbrace{(t e^{i2\pi/3})^3}_{t^3}+1} \cdot \underbrace{e^{i2\pi/3}\,dt}_{z'(t)\,dt}$$

$$= -e^{i2\pi/3} \int_0^R \frac{1}{t^3+1}\,dt \longleftarrow \text{want!}$$

Aha! $-L_2: t e^{i\alpha}$. Need $[e^{i\alpha}]^3 = 1$

Radius of conv of $\frac{\sin z}{z}$ about $z=\pi$?

removable. Give it value 1 at $z=0$.

entire! $\quad R=\infty$