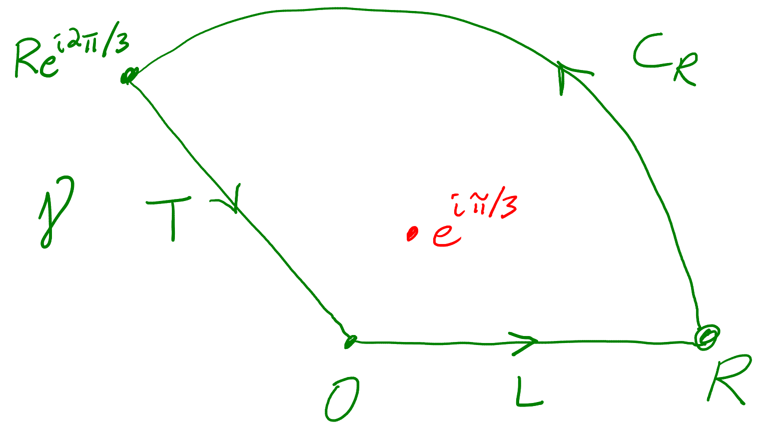
Review 2

Final exam Thurs, May 8, 8-10am, SC 239

No class Friday



$$\int_\gamma \underbrace{\frac{1}{z^3+1}}_{f(z)}\,dz$$

$$\int_0^\infty \frac{1}{x^3+1}\,dx$$

Show $\left|\int_{C_R} f\,dz\right| \to 0$ as $R\to\infty$.

$$|z^3+1| \geq \underbrace{\big|\,|z^3|-|1|\,\big|}_{\underbrace{|R^3-1| \text{ on } C_R}_{R^3-1 \text{ when } R>1}} \geq R^3-1 \quad \text{when } |z|=R>1.$$

So
$$\left|\int_{C_R}\frac{1}{z^3+1}\,dz\right| \leq \left(\max_{C_R}\left|\frac{1}{z^3+1}\right|\right)\text{Length}(C_R)$$

$$\leq \frac{1}{R^3-1}\cdot\left(\frac{2\pi}{3}R\right) \quad \text{when } R>1$$

$$\to 0 \quad \text{as } R\to\infty.$$

<u>or</u> Use Basic poly estimate. $\exists\, R_0>0$, $0<a<A$, with

$$\underbrace{a|z|^3 \leq |z^3+1|}_{\text{denom est.}} \leq A|z|^3 \quad \text{when } |z|>R_0.$$

Get $\left|\int_{C_R}\right| \leq \frac{1}{aR^3}\cdot\left(\frac{2\pi}{3}R\right)$ when $R>R_0$.

3. Let C_1 denote the unit circle parametrized in the counterclockwise sense. Evaluate the following integrals. Explain.

a) $\int_{C_1} \frac{e^{2z}}{z-4}\,dz = 0$ by Cauchy thm.

b) $\int_{C_1} \frac{e^{2z}}{4z^2+1}\,dz = 2\pi i \left(\mathrm{Res}_{i/2} + \mathrm{Res}_{-i/2} \right) = 2\pi i \left[\frac{e^{2(i/2)}}{8(i/2)} + \frac{e^{2(-i/2)}}{8(-\frac{i}{2})} \right]$

c) $\int_{C_1} \frac{e^{2z}}{2z^2-5z+2}\,dz \leftarrow$ quad formula.

$\mathrm{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}$ $\begin{cases} g(a)=0 \\ g'(a)\neq 0 \end{cases}$ simple zero

d) $\int_{C_1} \frac{e^{2z}}{(z-1/2)^5}\,dz$

e) $\int_{C_1} \frac{1}{z^5}\,dz = 0$

$\frac{d}{dz}\left[\frac{1}{-5+1} z^{-5+1}\right]$

Fund Thm Calc.

(Or Res Thm)

$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}}\,dz$

$n+1=5$

$n=4$

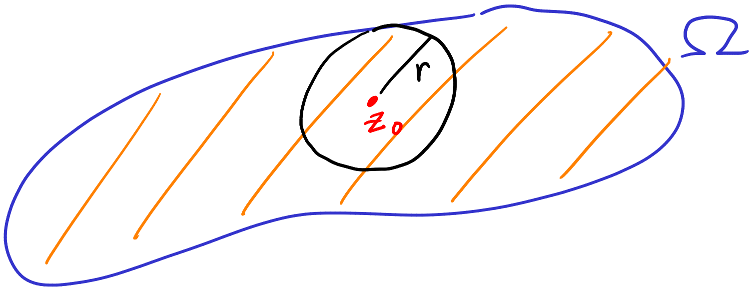
$I = \frac{2\pi i}{4!} \left[\frac{d^4}{dz^4} e^{2z} \right] \Big|_{z=\frac{1}{2}}$

$e^{2z} = A_0 + A_1(z-\tfrac{1}{2}) + A_2(z-\tfrac{1}{2})^2 + \cdots$

$\frac{e^{2z}}{(z-\frac{1}{2})^5} = \frac{A_0}{(z-\frac{1}{2})^5} + \cdots + \frac{A_4}{z-\frac{1}{2}} + \cdots$ Res

$\frac{1}{z^5} = \frac{1}{z^5} + \frac{0}{z^4} + \cdots + \frac{0}{z} + 0 + 0\cdot z + \cdots$ Res

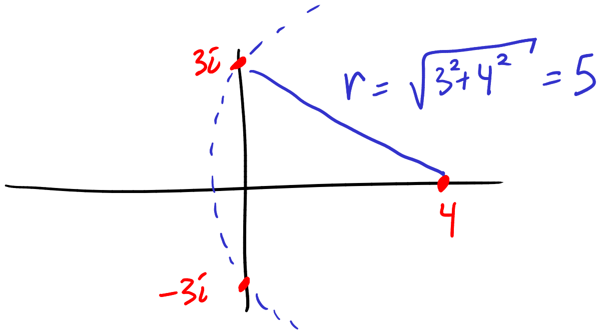
4. What is the radius of convergence of the power series for $\dfrac{\sin z}{z^2+9}$ centered at $z = 4$? Explain.



$f(z)$ analytic on Ω

Radius of conv of power series for f about $z_0 \in \Omega$ is $\geq r$

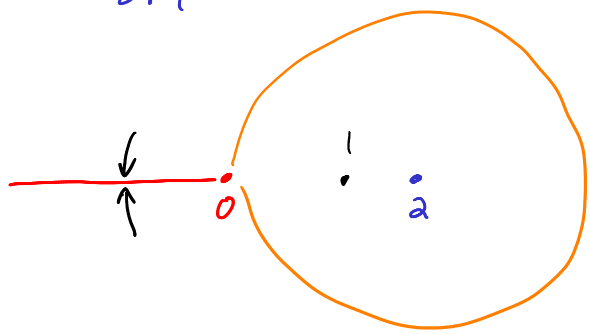
$\dfrac{\sin z}{z^2+9}$



$RoC \geq 5$

and $R \leq 5$ because of poles of $\dfrac{\sin z}{z^2+9}$ at $\pm 3i$.

EX $\dfrac{\operatorname{Log} z}{z-1}$ about $z_0 = 2$



$R = 1$ or $\underline{\underline{2}}$?

ans

$$\left.\begin{aligned} &\operatorname{Log} 1 = \operatorname{Ln} 1 + i \cdot 0 = 0 \\ &\frac{d}{dz}(\operatorname{Log} z)\Big|_{z=1} = \frac{1}{z}\Big|_{z=1} = 1 \neq 0 \end{aligned}\right\}$$ 1 is a simple zero of $\operatorname{Log} z$

$\dfrac{\operatorname{Log} z}{z-1}$ has a removable sing at $z=1$.

L'H rule $\dfrac{f(z)}{g(z)} = \dfrac{(z-z_0)^N F(z)}{(z-z_0)^N G(z)} \xrightarrow[z \to z_0]{} \dfrac{F(z_0)}{G(z_0)}$ not zero when f, g both have zeroes of order N at z_0.

$$= \frac{[f^{(N)}(z_0)/N!]}{[g^{(N)}(z_0)/N!]} =$$

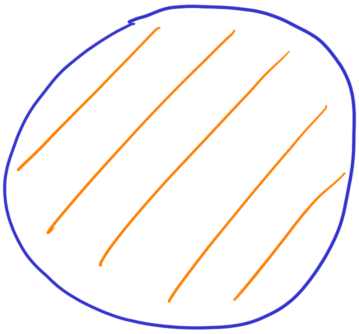
6. How many zeroes does the polynomial $\underbrace{z^4}_{\text{big } f(z)} + \overbrace{z^3 - z^2 + 2}^{h(z)}$ have inside the disc of radius two about the origin? Explain.

Rouché's

$|h(z)| < |f(z)|$ on γ.

Then f and $f+h$ have same # zeroes inside γ.

C_2



on C_2: $f(z) = z^4$ $\qquad |f(z)| = 2^4 = 16$ on C_2.

$h(z) = z^3 - z^2 + 2 \qquad |h(z)| \leq 2^3 + 2^2 + 2 = 14 < 16$

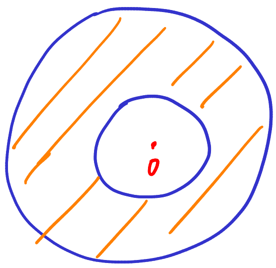
$|h(z)| \leq |z^3| + |-z^2| + |2|$

Get $|h(z)| < |f(z)|$ on C_2.

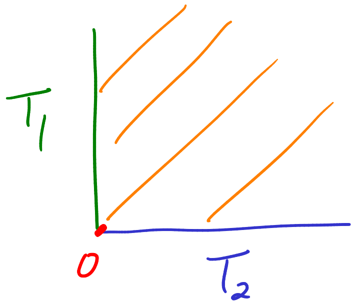
!

So poly has 4 zeroes counted with mult in C_2.

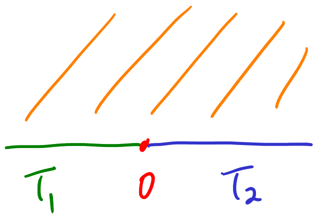
8. Find a harmonic function on the unit disc that has boundary values on the unit circle equal to one in the first quadrant, two in the second quadrant, three in the third quadrant, and four in the fourth quadrant.

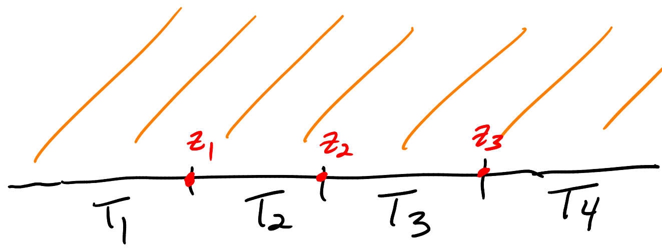


$$u = A\,\mathrm{Ln}|z| + B$$

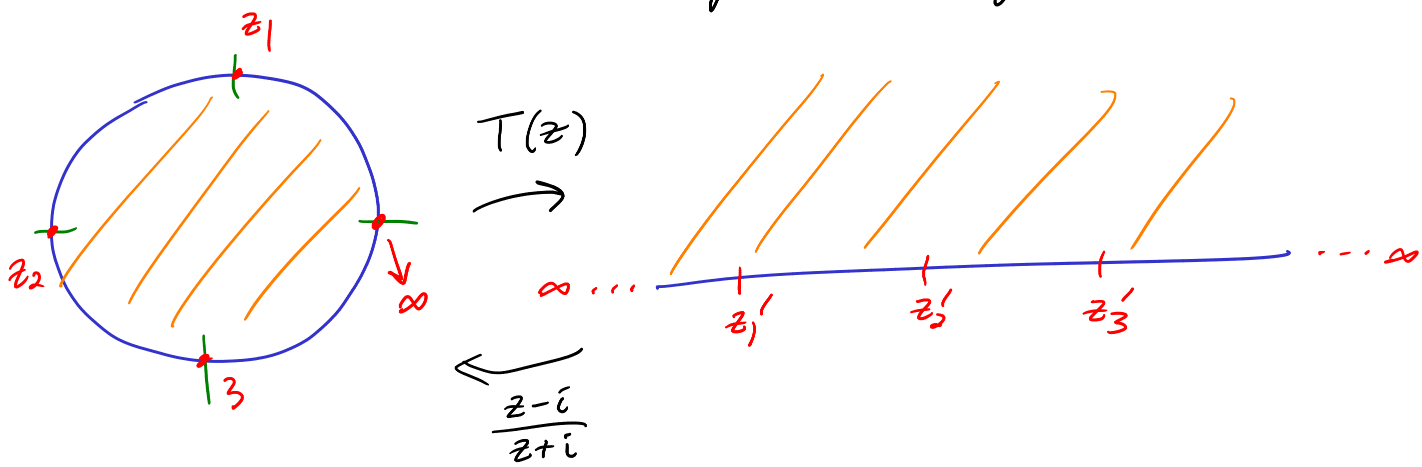


$$u = A \cdot \mathrm{Arg}\, z + B$$

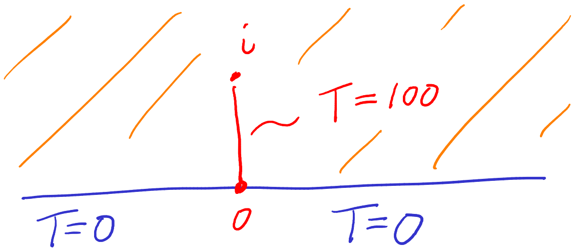




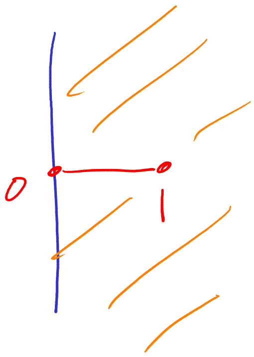
$$u = A\,\mathrm{Arg}(z-z_1) + B\,\mathrm{Arg}(z-z_2) + C\,\mathrm{Arg}(z-z_3) + D$$



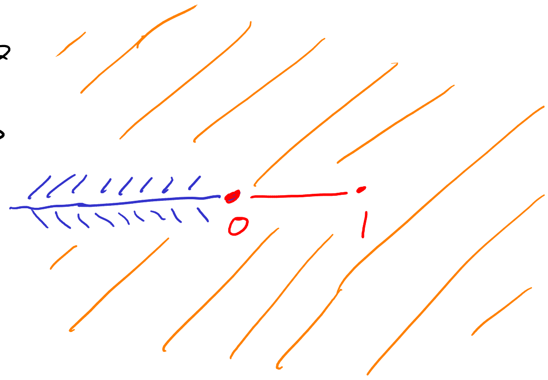
Heat problem



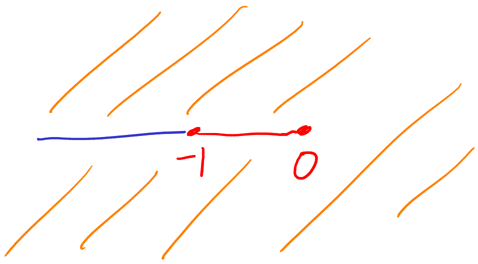
$$\xrightarrow[= -iz]{e^{-i\pi/2}\cdot z}$$



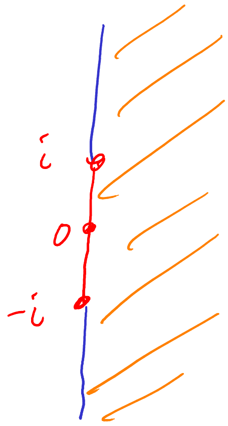
$$\xrightarrow{z^2}$$



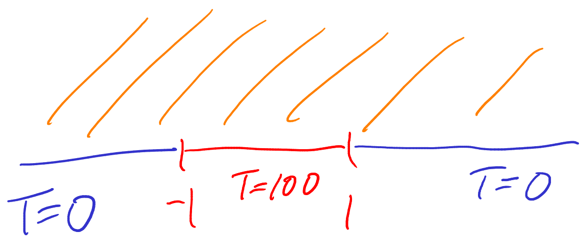
$$\xrightarrow{z-1}$$



$$\xrightarrow{z^{1/2}} \; = e^{\frac{1}{2}\operatorname{Log} z}$$



$$\xrightarrow{e^{i\pi/2} z = iz}$$



$$u = A\cdot \operatorname{Arg}(z+1) + B\cdot \operatorname{Arg}(z-1) + C$$

Find A, B, C

Sol^n : $u(f(z))$ where f = composition of analytic mappings