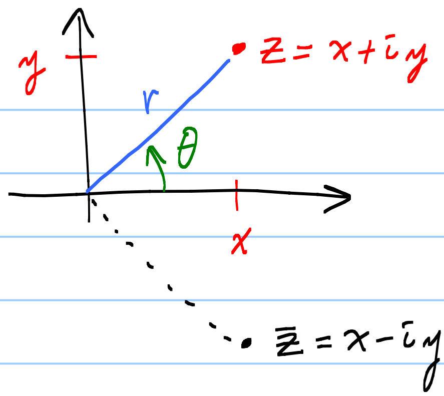
Lesson 2 MA 425 / MA 525



$x = \text{Re } z$ ← real part

$y = \text{Im } z$ ← imag part

modulus of z → $|z| = r = \sqrt{x^2+y^2}$

$\theta = \underline{\text{Arg}}\ z$ ← Principal argument

$-\pi < \theta \le \pi$

$\arg z = \{\theta_0 + 2\pi n : n \in \mathbb{Z},\ \theta_0 = \text{Arg } z\}$, a set

$\arg z$ = a choice of argument

Easy facts:

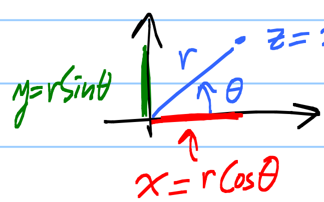
$\overline{(zw)} = \bar{z}\cdot\bar{w}$ $\qquad |zw| = |z|\cdot|w|$

$\overline{(z/w)} = \bar{z}/\bar{w}$ $\qquad |\bar{z}| = |z|$

$|z/w| = |z|/|w|$

Cartesian form: $z = x+iy$ great for $+$ ← just like vectors in $\mathbb{R}^2$

Polar form:



$z = x+iy = r\cos\theta + i r\sin\theta$

$= r\,(\underbrace{\cos\theta + i\sin\theta}_{e^{i\theta}}) = re^{i\theta}$

great for mult, roots, powers.

$$z^n = (re^{i\theta})^n = r^n\,(\underbrace{e^{i\theta}\cdot e^{i\theta}\cdots e^{i\theta}}_{n \text{ times}}) = r^n e^{in\theta}$$

De Moivre's Formula: $(e^{i\theta})^n = e^{in\theta}$

Why: $e^{i\alpha}\cdot e^{i\beta} = (\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta)$

$$= [\underbrace{\cos\alpha\cos\beta - \sin\alpha\sin\beta}_{\cos(\alpha+\beta)}] + i\,[\underbrace{\cos\alpha\sin\beta + \sin\alpha\cos\beta}_{\sin(\alpha+\beta)}]$$

$$= e^{i(\alpha+\beta)}$$

Complex exponential : $E(x+iy) = e^x \cos y + i\, e^x \sin y$

Consequence : $E(z+w) = E(z)\,E(w)$

Easy facts :

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

$$= \overline{e^{i\theta}}$$

$$\frac{1}{e^{i\theta}} = \frac{\overline{e^{i\theta}}}{|e^{i\theta}|^2} = e^{-i\theta}$$

Prob : $\int \sin^3\theta \, d\theta$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad (A)$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \quad (B)$$

A+B : $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

A−B : $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

$$\sin^3\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^3 = \frac{(e^{i\theta})^3 - 3(e^{i\theta})^2(e^{-i\theta}) + 3(e^{i\theta})(e^{-i\theta})^2 - (e^{-i\theta})^3}{8i^3}$$

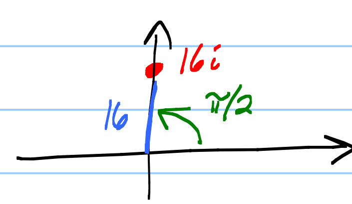
$$= \frac{e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}}{-8i}$$

$$= \frac{e^{i3\theta} - e^{-i3\theta}}{(-4)\cdot 2i} - 3\,\frac{e^{i\theta} - e^{-i\theta}}{(-4)\cdot 2i}$$

$$= -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$$ ← easy to integrate!

Finding roots : $z^4 = 16i$

$$(re^{i\theta})^4 = 16e^{i\pi/2}$$

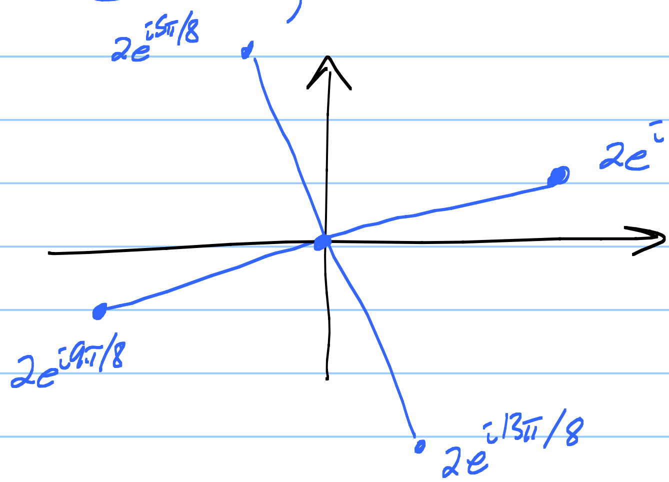


$$r^4 e^{i4\theta} = 16 e^{i\pi/2}$$

$r^4 = 16$ $\qquad$ $4\theta = \frac{\pi}{2} + 2\pi k$

$\boxed{r = 2}$ $\qquad$ $\boxed{\theta = \frac{\pi}{8} + k\frac{\pi}{2}}$ $\qquad$ $k = 0, \pm 1, \pm 2, \dots$

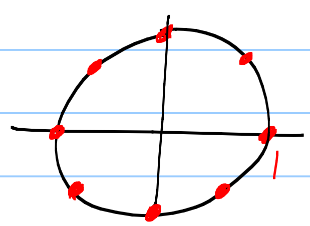
Roots: $z = 2e^{i\left(\frac{\pi}{8} + k\frac{\pi}{2}\right)}$, $k \in \mathbb{Z}$.



Only 4 4-th roots!

N-th roots of unity: $z^N = 1 = 1 \cdot e^{i0}$

$z = e^{ik2\pi/N}$ $\qquad$ $k = 0, \dots, N-1$



8-th roots of 1

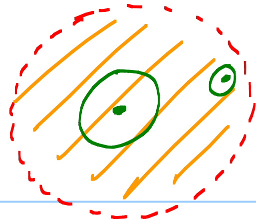
Topology of $\mathbb{C} \approx \mathbb{R}^2$: $\qquad$ $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

(Metric space) $\qquad$ $= \text{dist}_{\mathbb{R}^2}(z_1, z_2)$

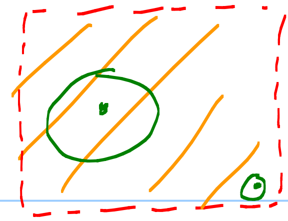
$D_R(z_0) = \{z : |z - z_0| < R\}$ ← disc of radius R about z_0

Def^n: $\Omega \subset \mathbb{C}$ is open if, for each point $z_0 \in \Omega$, there is a radius $r > 0$ such that $D_r(z_0) \subset \Omega$.

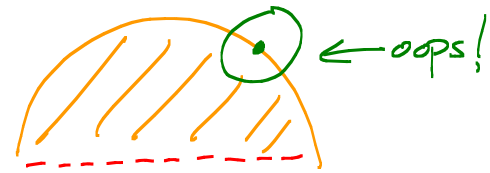
Note: $r = r(z_0)$



open



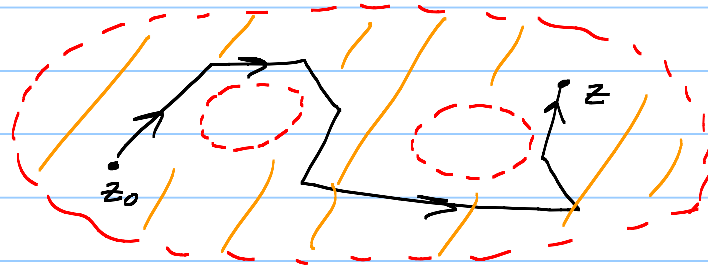
open



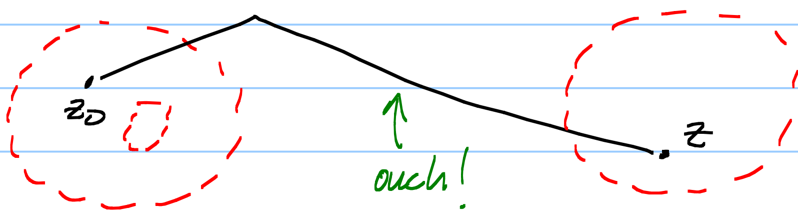
not open

Def^n: An open set Ω is <u>connected</u> if any two points in Ω can be connected by polygonal path that stays in Ω.

Fact: Can replace "polygonal" by "continuous."

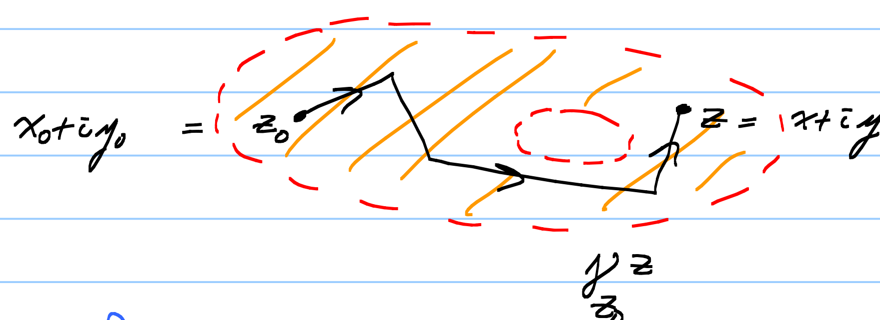


Feeling: One blob.



Not connected

Lemma: Suppose $u(x,y)$ is continuously diff'ble on an open connected set Ω and $\nabla u \equiv 0$ on Ω. Then u is constant on Ω.



Pf: $u(x,y) - u(x_0,y_0) = \int_{\gamma_{z_0}^{z}} \underbrace{\nabla u}_{0} \cdot d\vec{s} = 0$

Def^n: An open connected set in $\mathbb{C}$ is called a <u>domain</u> (or a <u>region</u>).