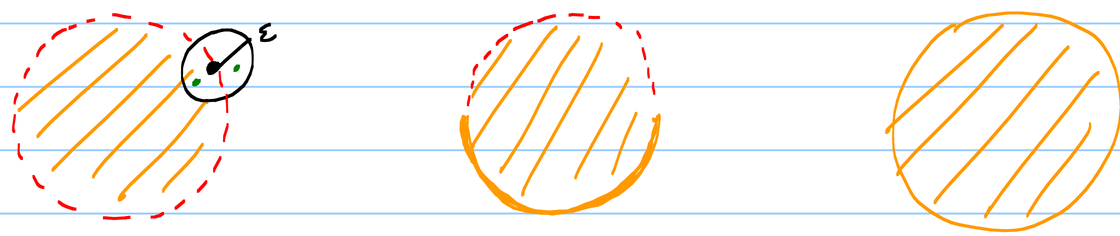
# Lesson 3

Open set $\Omega$: For each $z_0 \in \Omega$, there is a disc about $z_0$ completely contained in $\Omega$ : $r>0$ such that $D_r(z_0) \subset \Omega$.

Connected open set $\Omega$: Any two points in $\Omega$ can be connected by a polygonal path completely contained in $\Omega$.

Def$^n$: A domain is an open connected set in $\mathbb{C}$.

Def$^n$: A point $z_0$ is a boundary point of a set $S \subset \mathbb{C}$ if for every $\varepsilon > 0$ (no matter how small), $D_\varepsilon(z_0) - \{z_0\}$ contains a point in $S$ and a point in $\mathbb{C} - S$.



open

not open
not closed

not open
"closed"

Def$^n$: A set is closed in $\mathbb{C}$ if it contains all its boundary pts.

Fact: $S$ is open $\Longleftrightarrow$ $\mathbb{C} - S$ is closed

## Calculus in $\mathbb{C}$:

Limits : Sequence $\{z_n\}_{n=1}^{\infty}$ converges to $z_0$ means :

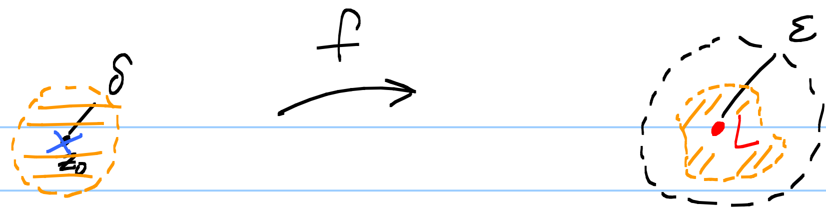
given any $\varepsilon > 0$, there is an $N$ such that

$|z_n - z_0| < \varepsilon$ when $n > N$. Write $\lim\limits_{n\to\infty} z_n = z_0$.

Function : $\lim\limits_{z \to z_0} f(z) = L$ means :

given $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $|z - z_0| < \delta$ and $z \neq z_0$.

**Note:** $f$ does not even need to be defined at $z_0$.




**Def$^n$:** $f$ is continuous at $z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

**EX:** $f(z) = z^2$

$$DQ = \frac{f(z) - f(a)}{z - a} = \frac{z^2 - a^2}{z - a} = \frac{(z-a)(z+a)}{z-a} = z + a \longrightarrow 2a \text{ as } z \to a.$$

not defined at $z = a$

$\lim_{z \to a} DQ = 2a$ : Let $\varepsilon > 0$. $|DQ - 2a|$

$$= |(z+a) - 2a| = |z - a| < \varepsilon$$

want

Aha! Can take $\delta = \varepsilon$.

Exercise: Show $\lim_{z \to a} z^2 = a^2$ using $\varepsilon, \delta$ def$^n$.

**Two important ineq:**

$|z + w| \leq |z| + |w|$ ($\Delta$ ineq) ← numerator estimate

$|z - w| \geq \big||z| - |w|\big|$ ← denominator estimate

$$|z+w| = |z - (-w)| \geq \underbrace{\big||z| - |-w|\big|}_{\big||z| - |w|\big|}$$

**Basic facts of limits** $\mathbb{C} \to \mathbb{C}$ hold same as $\mathbb{R} \to \mathbb{R}$ with same proofs!

If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then

1) $\lim_{z \to z_0} [f(z) + g(z)] = A + B$

2) $\lim_{z \to z_0} [f(z)\, g(z)] = A \cdot B$

3) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$ provided that $B \neq 0$.

Facts: 1) $f(z) \equiv c$, a const. is continuous on $\mathbb{C}$

2) $f(z) = z$ is cont. $(\varepsilon = \delta)$

3) Hmmm. $z^2 = z \cdot z$. Product of 2 cont fcns is cont.

$z^3$ etc. ...

4) complex polynomials are continuous

5) rational functions are too at points where denom $\neq 0$.

Def$^n$: $f(z)$ is complex diff'ble at $z_0$ if

$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. Call it $f'(z_0)$ if it does.

Remark: Same as $\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$. $(\Delta z = z - z_0)$

MA 425/525: Complex diff'ble fcns on a domain.

Fact: Differentiation facts $\mathbb{C} \to \mathbb{C}$ are exactly the same as $\mathbb{R} \to \mathbb{R}$ with same proofs!

1) $(f+g)' = f' + g'$

2) $(fg)' = f'g + fg'$

3) $(f/g)' = \frac{f'g - fg'}{g^2}$ where $g$ not zero.

Fact: Complex polys are complex diff'ble. Rational fcns too where denom not zero.

$f(z) \equiv c$ : $DQ = \frac{f(z) - f(a)}{z - a} = \frac{c - c}{z - a} \equiv 0$

$f(z) = z$ : $DQ = \frac{z - a}{z - a} = 1$

$f(z) = z^2$ :  $DQ = \frac{z^2 - a^2}{z - a} = z + a \to 2a$  $f'(a) = 2a$

or $f'(z) = 2z$.

Use induction and product rule to get:

$z^3 = z^2 \cdot z$  $(z^3)' = (2z) \cdot z + (z^2) \cdot 1 = 3z^2$

$\ldots$

$z^n = z^{n-1} \cdot z$

$\ldots$

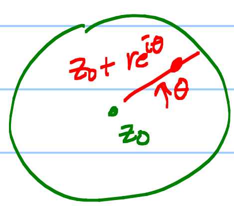
Induction shows: $f(z) = z^n$. Then $f'(z) = n z^{n-1}$

EX: $f(z) = \bar{z}$ is not complex diff'ble

$f(x+iy) = x - iy$

$DQ = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0}$  Hmmm: $z = z_0 + re^{i\theta}$ ← r small

$= \frac{\overline{(z_0 + re^{i\theta})} - \bar{z}_0}{(z_0 + re^{i\theta}) - z_0} = \frac{r\overline{e^{i\theta}}}{re^{i\theta}} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}$ ← ouch!




$DQ \longrightarrow e^{-2i\theta}$