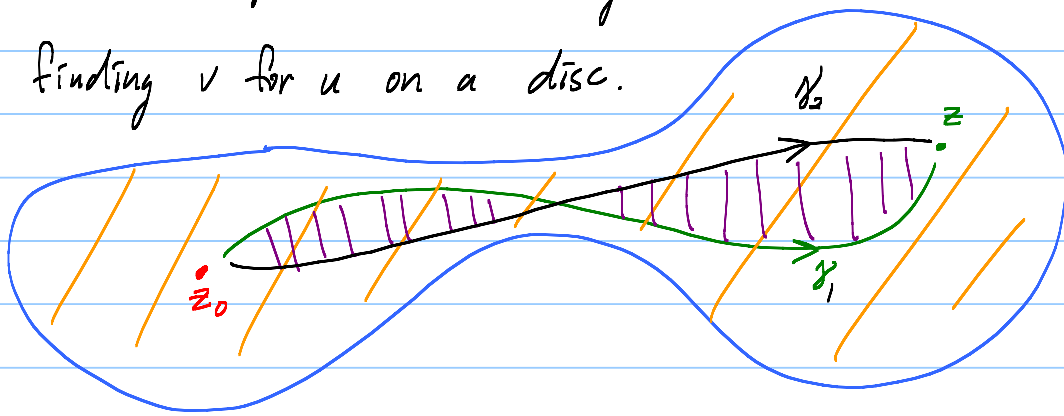
Lesson 8 Harmonic conjugates, complex exponential   7, 8 due Wed.

$f = u + iv$ $\qquad$ $\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix} \quad \longrightarrow \quad \begin{matrix} v_x = -u_y \\ v_y = u_x \end{matrix}$ $\qquad$ $\nabla v = \begin{pmatrix} -u_y \\ u_x \end{pmatrix}$

Last time: finding v for u on a disc.



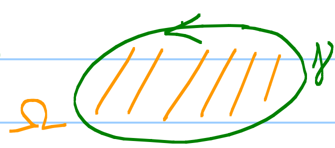
Ω domain with no holes

$$v(z) - v(z_0) = \int_\gamma \nabla v \cdot d\vec{s} = \int_\gamma \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \int_\gamma -u_y\,dx + u_x\,dy$$

Hmmm. "Define" v this way and try to show it works.

Step 1: Show well defined. $\left(\int_{\gamma_1} - \int_{\gamma_2}\right) = \int_{\text{closed loops}}$

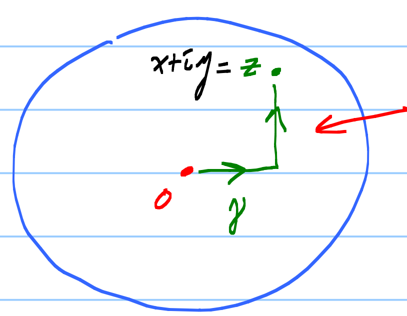
Green's Theorem:



$$\int_\gamma \underbrace{P}_{-u_y}dx + \underbrace{Q}_{u_x}dy = \iint_\Omega \underbrace{\underbrace{\frac{\partial Q}{\partial x}}_{u_{xx}} - \underbrace{\frac{\partial P}{\partial y}}_{-u_{yy}}}_{\Delta u \equiv 0}\,dx\,dy \quad !$$

Step 2: Show $v_x = -u_y$.
$v_y = u_x$

Back to $\Omega = D_1(0)$. $z_0 = 0$



$x(t) = x$
$y(t) = t$
$0 \le t \le y$

$$\underset{\text{define}}{v} = \int_\gamma -u_y\,dx + u_x\,dy$$

Step 1: Show $v_y = u_x$.

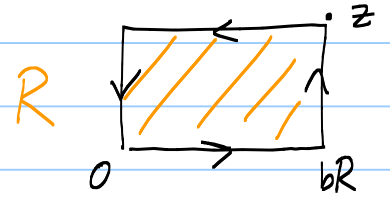
$$v = \underbrace{\int_{0 \to x+i0}}_{\text{const}} + \int_{x+i0 \to x+iy}$$

$$V = (\text{const}) + \int_0^y -u_y(x,t)\cdot \underset{\substack{\uparrow\\ \frac{dx}{dt}}}{0}\, dt + u_x(x,t)\cdot \underset{\substack{\uparrow\\ \frac{dy}{dt}}}{1}\, dt$$

$$V = (\text{const}) + \int_0^y u_x(x,t)\, dt$$

Aha! Fund. Thm. Calc. : $\dfrac{\partial v}{\partial y} = u_x(x,y)$ ✓

Step 2:



Green's Thm: $\displaystyle\int_{bR} -u_y\,dx + u_x\,dy = \iint_R \Delta u\, dx\,dy = 0$

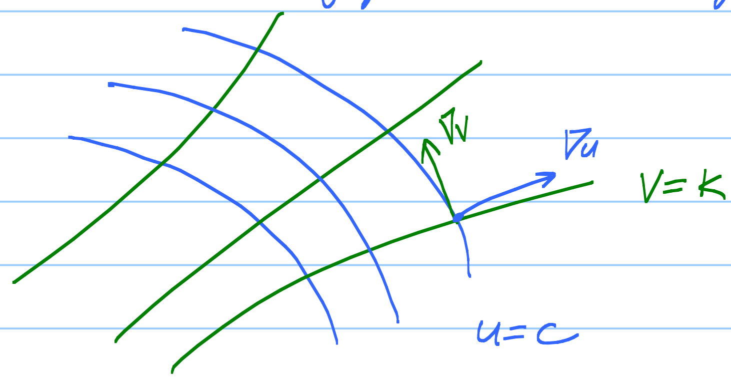
Aha! $\displaystyle v = \int = \int$

Step 3: Use similar argument to show that $\dfrac{\partial v}{\partial x} = \dfrac{\partial}{\partial x}\displaystyle\int = -u_y$

Complex chain rule: f, g analytic and $h(z) = f(g(z))$, then

$$h'(z) = f'(g(z))\, g'(z).$$

Read 2.6: Fact: level sets of a harmonic function u and its harmonic conjugate are orthogonal!



$$\nabla u \cdot \nabla v = \begin{pmatrix} u_x \\ u_y \end{pmatrix} \cdot \underset{\substack{\uparrow \\ \text{C-R} \\ \text{Eqns}}}{\begin{pmatrix} -u_y \\ u_x \end{pmatrix}} = 0$$

Complex exponential: Look for an analytic function $E(z)$ such that $E(0) = 1$ and $E'(z) = E(z)$.

Fun game: Hmmm. $E(x+iy) = u(x,y) + i\,v(x,y)$

$E(0)=1$: $u(0,0)=1$, $v(0,0)=0$

$$E'(z) = \begin{cases} u_x + i v_x \overset{\text{want}}{=} u + iv \\ \underbrace{-v_y + i u_y}_{-k'(y)e^x + i\,C'(y)e^x} \end{cases}$$

Easy "ODE"s

$u_x = u \leftarrow u = C(y)e^x$

$v_x = v \leftarrow v = k(y)e^x$

$$\boxed{u+iv = C(y)e^x + i k(y)e^x}$$

$-k'(y)e^x + i\,C'(y)e^x$

$\| \leftarrow$ want

$u+iv = C(y)e^x + i\,k(y)e^x$

Need $\begin{cases} C(y) = -k'(y) & (A) \\ k(y) = C'(y) & (B) \end{cases}$

$(B)'$: $k'(y) = C''(y)$

Plug this in (A): $C(y) = -C''(y)$ $\quad \boxed{C'' + C = 0}$

Similarly $\boxed{k'' + k = 0}$

Finally, initial conditions show $C(y) = \cos y$, $k(y) = \sin y$.

Define $E(x+iy) = e^x \cos y + i\, e^x \sin y$

C-R Eqns hold. So E is analytic.

Properties:

1) $E(0) = 1$
2) $E'(z) = E(z)$

pin down $E(z)$

3) $E(z) \neq 0$ for all z
4) $E(z+w) = E(z)\cdot E(w)$
5) $E(z-w) = E(z)/E(w)$
6) $E(-z) = 1/E(z)$

7) $E(z) = 1 \iff z = 2\pi n i, \ n \in \mathbb{Z}$

Lemma: If $f(z)$ is analytic and $f'(z) = k f(z)$, then

$$f(z) = c E(kz) \quad \text{where} \quad c = f(0).$$

Why: $f' = kf$: $f' - kf \equiv 0$

[Math 262
Int. factor
$\mu = e^{\int -k\,dx} = e^{-kx}$
Hmmm. Mult by
$E(-kz)$]

$$E(-kz) f'(z) - k \underbrace{E(-kz)}_{E'(-kz)} f(z) \equiv 0$$

$-k = \frac{d}{dz}(-kz)$

$$\frac{d}{dz}\Big[E(-kz) f(z) \Big] \equiv 0$$

Aha! $E(-kz) f(z) \equiv \text{const.}$

Plug in $z=0$ to see const $= f(0)$.

So $f(z) = c / E(-kz)$ where $c = f(0)$

Hmmm. Is $E(-kz) = 1/E(kz)$? Yes.