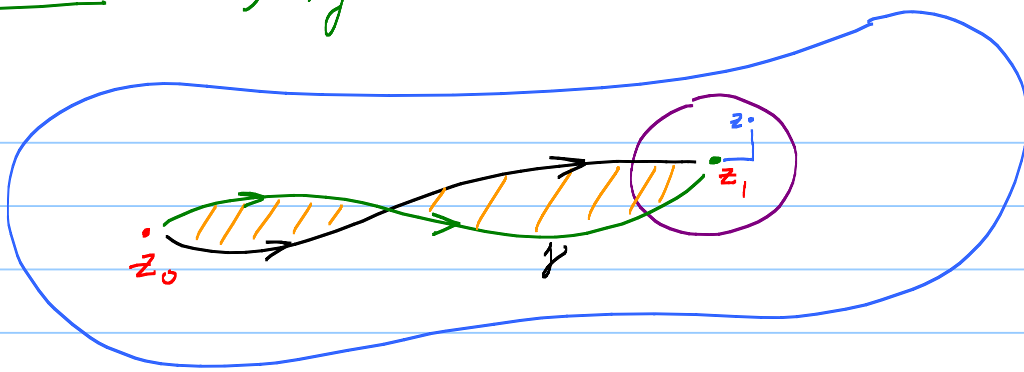
Lesson 9 e^z, $\log z$



Harmonic conjugate v for u, $\Delta u \equiv 0$

$v(z) = \int_\gamma -u_y\,dx + u_x\,dy$ <u>well defined</u> by Green's Thm

stick disc on end of γ and do zig, zag and zag, zig thing.

<u>Lemma</u>: If f satisfies $\boxed{f'(z) = k f(z)}$ (where $k \in \mathbb{C}$) with $f(0) = c$, then $f(z) = c\,E(kz)$.

<u>Pf</u>: Last time: showed that $\underline{f(z) = c / E(-kz)}$ where $c = f(0)$.

Hmmm. Notice that $f(z) = E(kz)$ satisfies conditions of the lemma: $\frac{d}{dz} E(kz) = E'(kz) \frac{d}{dz}(kz) = E(kz) \cdot k$

Aha! So $\underbrace{f(z)}_{E(kz)} = c / E(-kz)$ $\quad c = E(k \cdot 0) = 1$

So $E(-kz) = 1/E(kz)$. Back to letting $f(z)$ be $f(z)$

So $f(z) = c\,E(kz)$ where $c = f(0)$. ✓

<u>Prop</u>:
1) $E(0) = 1$ ✓
2) $E'(z) = E(z)$ ✓
3) $E(z) \neq 0$ for all z ⟵ follows from $E(z)E(-z) = 1$
4) $E(z+w) = E(z)E(w)$
5) $E(-z) = 1/E(z)$ ⟵ came from lemma
6) $E(z-w) = E(z)/E(w)$ ⟵ follows from 4, 5
7) $E(z) = 1 \iff z = 2\pi n i \quad n \in \mathbb{Z}$

Pf of 4: $f(z) = E(z+w)$ where w is held fixed.

$$f'(z) = E'(z+w)\cdot \frac{d}{dz}[z+w] = E(z+w)\cdot(1+0)$$

$$= 1\cdot f(z)$$

Lemma $\Rightarrow$ $f(z) = c\,E(1\cdot z)$ where $c = f(0) = E(0+w)$

Aha! $E(z+w) = E(w)\cdot E(z)$ ✓

7) $E(x+iy) = e^x(\cos y + i\sin y) = 1$

$|E(x+iy)| = e^x = 1 \Rightarrow e^x = 1$. So $x = \ln 1 = 0$

$x=0$. Also need $\begin{cases} \cos y = 1 \\ \sin y = 0 \end{cases}$ $\leftarrow$ $y = n\pi$ $n \in \mathbb{Z}$

But $\cos y = -1$ for (odd)π. $\cos y = 1$ for (even)π. ✓

Chain Rule: $h(z) = f(g(z))$. Then $h'(z) = f'(g(z))g'(z)$

Pf: f diff'ble: $\boxed{DQ = \dfrac{f(w)-f(A)}{w-A} = f'(A) + E(w)}$

where $E(w) \to 0$ as $w \to A$. Define $E(A) = 0$.

Next. Write $w = g(z)$ and $A = g(a)$

Solve DQ eqn for $f(w) - f(A)$:

$$f(w) - f(A) = f'(A)(w-A) + E(w)(w-A)$$

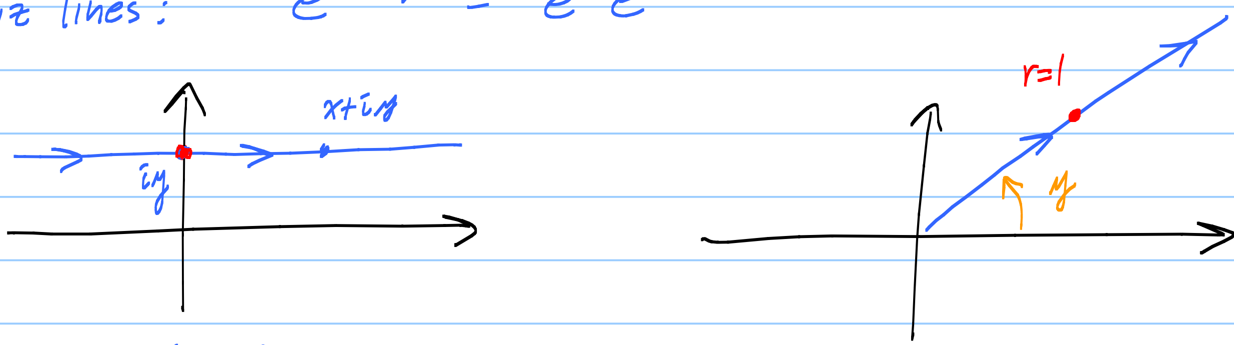
$$\left[\frac{f(g(z)) - f(g(a))}{z-a} = f'(g(a))\frac{[g(z)-g(a)]}{z-a} + E(g(z))\cdot\frac{g(z)-g(a)}{z-a}\right]$$

Let $z \to a$: $h'(a) = f'(g(a))\cdot g'(a) + \underbrace{E(g(a))}_{0}\cdot g'(a)$

Notation: From now on $e^{x+iy} = E(x+iy)$ ← the only analytic fcn that agrees with e^x on $\mathbb{R}$.

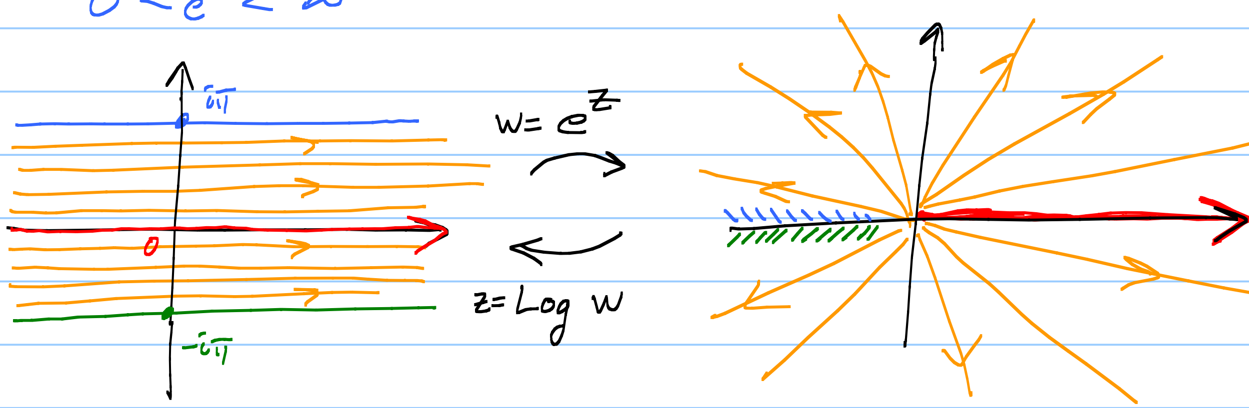
Graph of e^z:

Horiz lines: $e^{x+iy} = e^x e^{iy}$



$-\infty < x < \infty$

$0 < e^x < \infty$



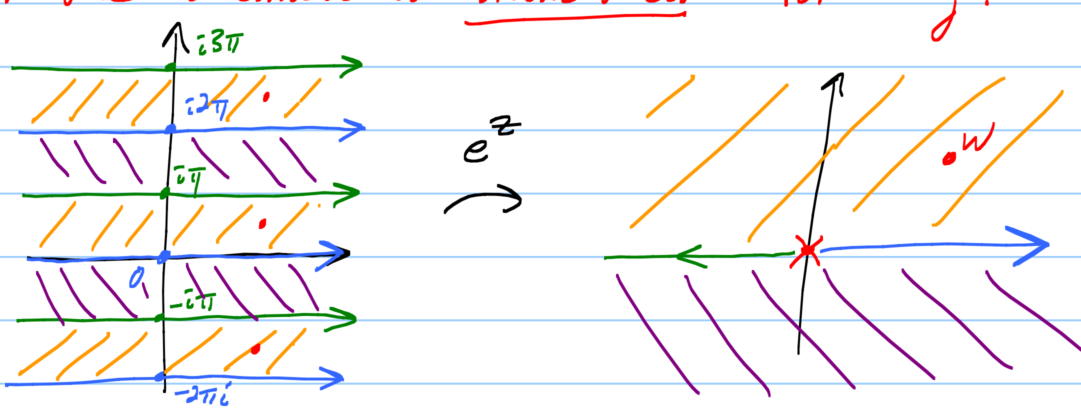
Principal branch of the complex log function:

$$\operatorname{Log} w = \operatorname{Ln}|w| + i \operatorname{Arg} w$$

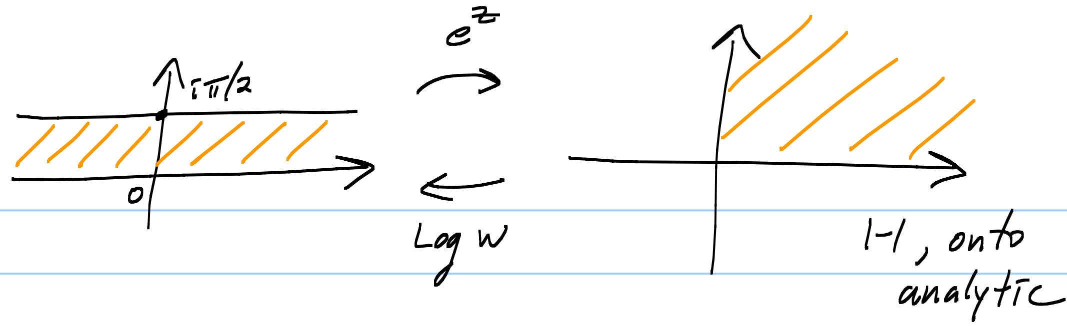
where $\operatorname{Arg} w = \theta$ such that $w = |w| e^{i\theta}$ with $-\pi < \theta \leq \pi$.

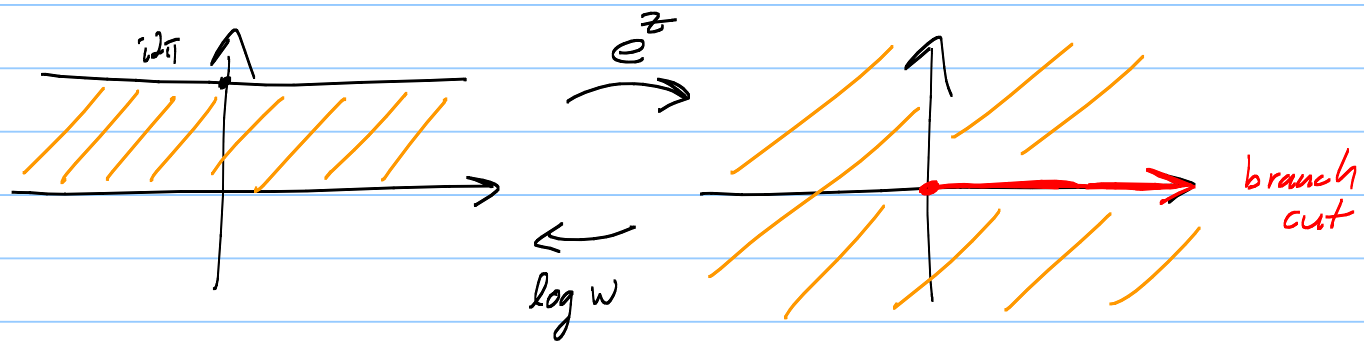
Negative real line is called a branch cut for Log.

Big picture:



Small pictures:





Branch log: $\log w = \operatorname{Ln}|w| + i\theta$ where $\theta \in \arg w$
with $0 < \theta < 2\pi$

<u>Note</u>: Ln y is uniquely well defined for real $y > 0$
but log w for complex w is <u>infinitely</u> valued!

<u>Fact</u>: Inverse of an analytic fcn must be analytic.