Lesson 11 Complex integration

9, 10, 11 due Wed.
Start reading Chap. 4

$z^p = e^{p \log z}$ can be infinitely valued!

$z^n = e^{n \log z} = e^{n(\text{Ln}|z| + i \arg z)} = e^{n \text{Ln}|z|} \underbrace{e^{i(n\theta + n 2\pi m)}}_{e^{in\theta} \cdot 1}$

Only one value for z^n! $= z^n$ if $n \in \mathbb{Z}$

$z^{p/q}$ has q distinct values.

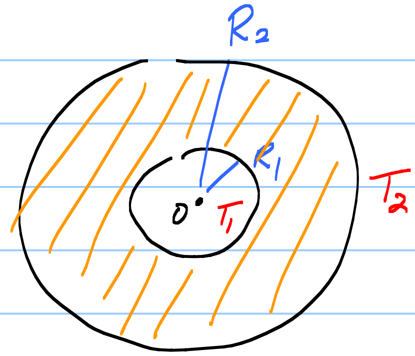
EX: $i^i = e^{i \log i} = e^{i(\underbrace{\text{Ln}|i|}_{0} + i(\frac{\pi}{2} + 2\pi m))} = e^{-(\frac{\pi}{2} + 2\pi m)}$

∞-many "i-th roots of i" some go to zero, some to ∞, all real and > 0!

3.4 washers, wedges, walls

$\text{Log}\, z = \underbrace{\text{Ln}|z|}_{\text{harmonic}} + i \underbrace{\text{Arg}\, z}$

Prob: Find steady state temp in a plate:



$\Delta T = c \frac{\partial T}{\partial t} \rightarrow 0$ at steady state

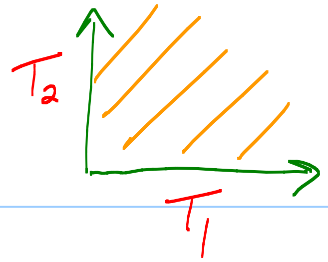
So T harmonic

Hmmm. $\text{Ln}|z|$ is harmonic and radially symmetric!

Aha! Try $T = A \,\text{Ln}|z| + B$

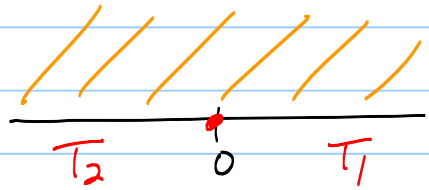
Need
$T_2 = A \,\text{Ln}\, R_2 + B$
$T_1 = A \,\text{Ln}\, R_1 + B$
$\leftarrow \det = \text{Ln} R_2 - \text{Ln} R_1 \neq 0$

Prob:



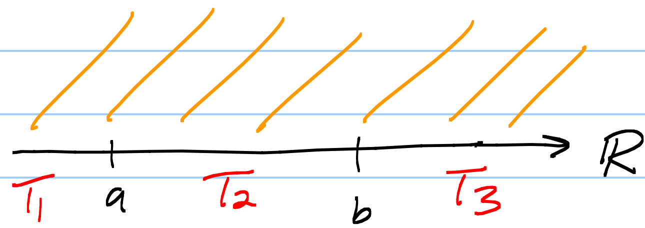
$$T = A \operatorname{Arg} z + B$$

Prob:



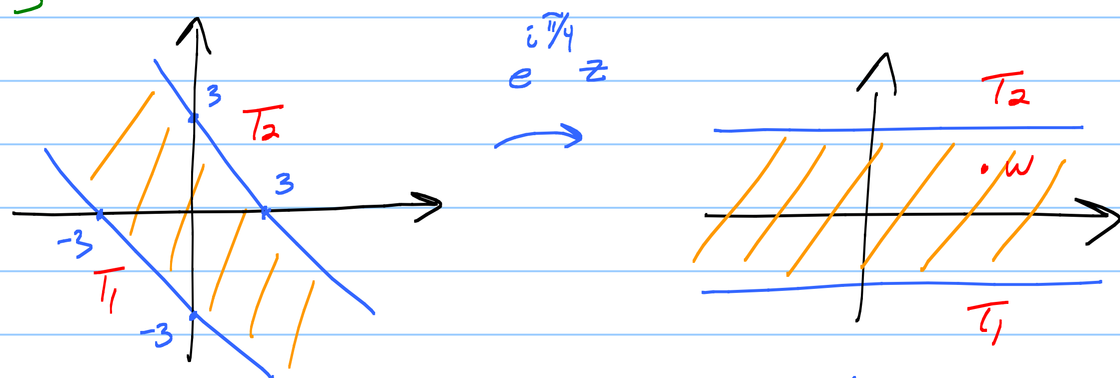
Same answer!

Prob:



$$T = A \operatorname{Arg}(z-a) + B \operatorname{Arg}(z-b) + C$$

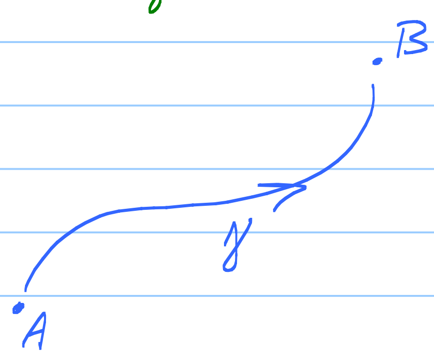
HWK 4: 3.4: 3



$$T = A \operatorname{Im} w + B$$

Complex integration: Want $\int_\gamma f(z)\, dz$?

Curves in $\mathbb{C} \approx \mathbb{R}^2$

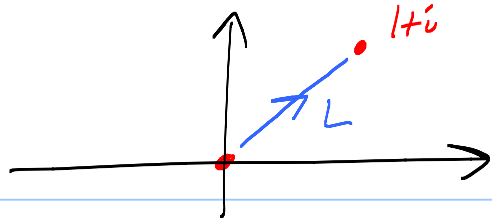


$\gamma : (x(t), y(t))$

$a \le t \le b$

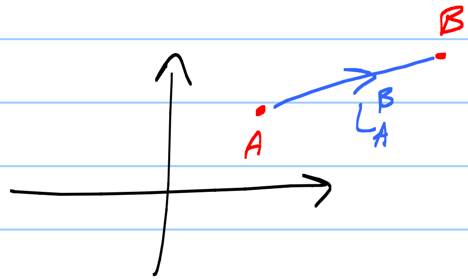
Better way: $\gamma: \; z(t) = x(t) + i\,y(t), \quad a \le t \le b$

EX:



$z(t) = t(1+i) \quad 0 \leq t \leq 1$

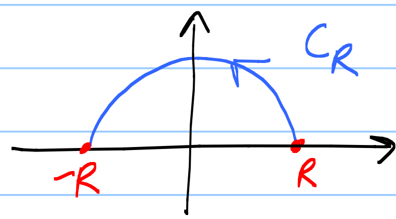
EX:



$L_A^B : \; z(t) = A + t(B-A)$

$0 \leq t \leq 1$

EX:



$C_R : \; z(t) = R e^{it}$

$0 \leq t \leq \pi$

Basic complex integral : Complex function of a real variable.

$$z(t) = x(t) + i y(t)$$

$$\int_a^b z(t)\, dt = \lim_{|\Delta t| \to 0} \underbrace{\sum_{k=1}^{N} z(t_k) \Delta t_k}_{= \sum x(t_k)\Delta t_k + i \sum y(t_k) \Delta t_k}$$

$$= \int_a^b x(t)\, dt + i \int_a^b y(t)\, dt$$

def^n makes sense if $z(t)$ is continuous.

Also $$z'(t) = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t} = \lim_{\Delta t \to 0} \left(\begin{matrix} DQ \\ \text{for } x(t) \end{matrix} \right) + i \left(\begin{matrix} DQ \\ \text{for } y(t) \end{matrix} \right)$$

$$= x'(t) + i y'(t)$$

def^n makes sense when $x(t), y(t)$ diff'ble

Fundamental Theorem of Calculus I:

$$\int_a^b z'(t)\,dt = z(b) - z(a)$$

$$\| $$

Why: $\displaystyle\int_a^b x'(t)\,dt + i\int_a^b y'(t)\,dt$ = Fund. Thm. Calc. of Freshman Calc. ✓

Basic inequality: $\displaystyle\left|\int_a^b z(t)\,dt\right| \le \int_a^b |z(t)|\,dt$

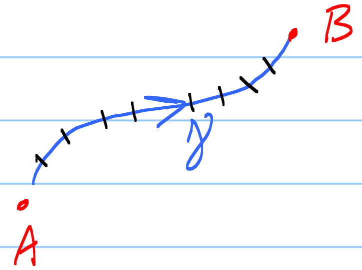
Why: $\displaystyle\left|\sum_{k=1}^{N} z(t_k)\,\Delta t_k\right| \le \sum_{k=1}^{N} |z(t_k)|\,\Delta t_k$

Now let $|\Delta t| \to 0$.

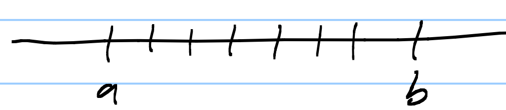
Lemma: $k \in \mathbb{C}$: $\displaystyle\int_a^b k\,z(t)\,dt = k\int_a^b z(t)\,dt$

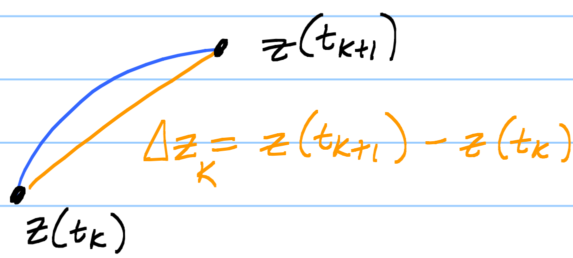
Pf: Multiply out, separate real and imag parts. ✓

Integration of a complex function along a curve in $\mathbb{C}$



γ: $z(t)$ $\quad a \le t \le b$



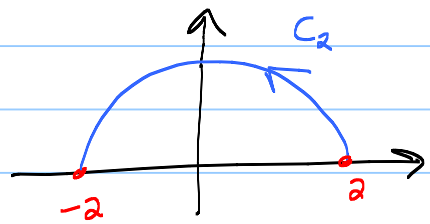


Hmmm. $$\int_\gamma f(z)\,dz = \lim_{|\Delta t|\to 0} \sum_{k=1}^{N} f(z(t_k))\, \underbrace{\Delta z_k}_{\approx\, z'(t_k)\,\Delta t_k}$$

$$= \int_a^b f(z(t))\, \underbrace{z'(t)\,dt}_{\text{think } dz = z'(t)\,dt}$$

Defⁿ: $$\int_\gamma f\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

EX:



$$C_2:\ z(t) = 2e^{it} = 2\cos t + i\,2\sin t$$
$$0 \le t \le \pi$$

$$z'(t) = 2(-\sin t) + i\,2\cos t$$
$$= 2i\,e^{it}$$

$$\int_{C_2} e^{z}\,dz = \int_0^{\pi} e^{z(t)}\, z'(t)\,dt$$

$$= \int_0^{\pi} e^{2\cos t + i\,2\sin t}\,\left(2i\,e^{it}\right) dt \longleftarrow \text{Uhg!}$$

Fund. Thm. Calc. II: $$\int_{\gamma_A^B} f'(z)\,dz = f(B) - f(A)$$