Lesson 13: Cauchy's Theorem

Fact: f has an analytic antiderivative on a domain Ω if and only if $\int_\gamma f\,dz = 0$ for all closed curves γ in Ω.

Pf: ($\Rightarrow$) easy. If $f = F'$, then $\int_\gamma f\,dz = \int_\gamma F'\,dz$

$$= \underbrace{F(\text{END}) - F(\text{START})}_{=} = 0$$

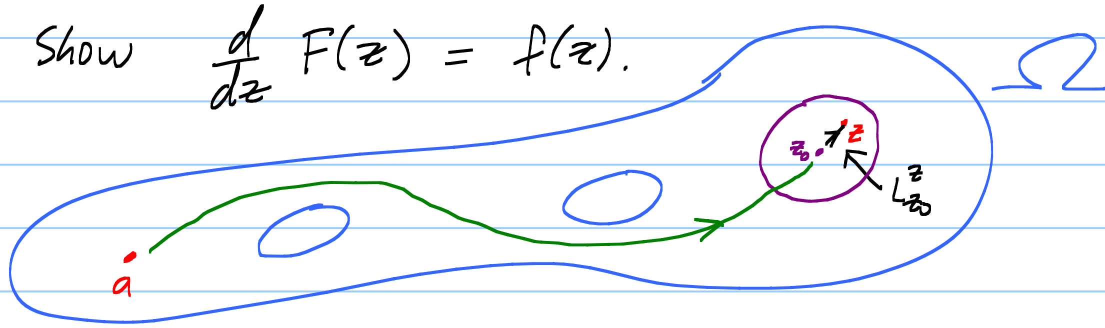
($\Leftarrow$): Define $F(z) = \int_{\gamma_a^z} f(w)\,dw$

Assume $\int_\gamma f\,dz = 0$ for all closed γ. ← Use this to see F well defined.

Lemma: $\int_{\gamma_a^z} c\,dw = c(z-a)$ because $c = \frac{d}{dz}[cz]$

$$= \int_{\gamma_a^z} \frac{d}{dw}[cw]\,dw = (cz) - (ca) = c(z-a) \checkmark$$

Step 2: Show $\frac{d}{dz} F(z) = f(z)$.



$$\gamma_a^z = \gamma_a^{z_0} \cup L_{z_0}^z$$

$$\frac{d}{dz} F(z) = \frac{d}{dz}\left(\underbrace{\int_{\gamma_a^{z_0}} f\,dw}_{=\text{const.}} + \int_{L_{z_0}^z} f\,dw \right)$$

f analytic $\Rightarrow$ f continuous. So near z_0:

$f(z) = f(z_0) + E(z)$ where $E(z) \to 0$ as $z \to z_0$.

$$DQ = \frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\int_{L_{z_0}^{z}} \underbrace{f(w)}_{f(z_0)+E(w)}\,dw$$

$$= \frac{1}{z-z_0}\left[f(z_0)(z-z_0) + \int_{L_{z_0}^{z}} E(w)\,dw \right]$$

$$= f(z_0) + \underbrace{\frac{1}{z-z_0}\int_{L_{z_0}^{z}} E(w)\,dw}_{\mathcal{E}(z)}$$

Finally $|\mathcal{E}(z)| \le \frac{1}{|z-z_0|}\left(\max_{w\in L_{z_0}^{z}} |E(w)|\right) \underbrace{\text{Length}\,(L_{z_0}^{z})}_{|z-z_0|}$

$\longrightarrow 0$ because $E(w)\to 0$ as $w \to z_0$.

Hmmm: Polynomials always have analytic antiderivatives.

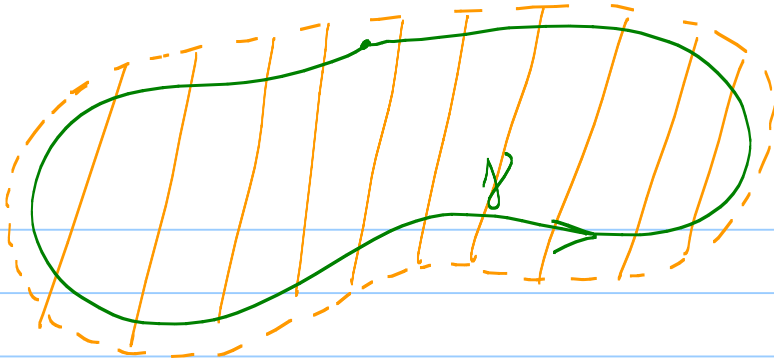
$$\left[\quad a_n z^n = \frac{d}{dz}\left[\frac{1}{n+1} a_n z^{n+1}\right] \quad \right]$$

So $\int_\gamma \text{poly}\, dz = 0$ for any closed γ.

Hmmm: $\int_\gamma f\,dz = 0$ if $f =$ Limit of polys (think power series).

Read 4.4b instead of 4.4a.

Cauchy's Theorem: Suppose f is analytic "inside and on" a closed Jordan curve γ. Then $\int_\gamma f\,dz = 0$.



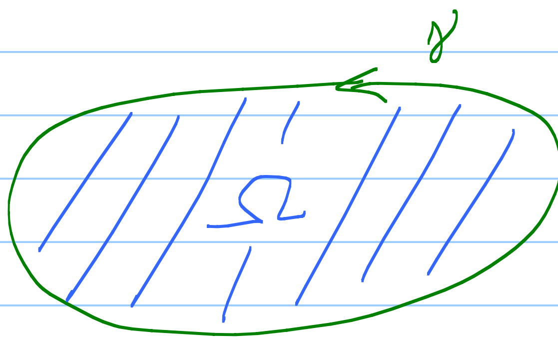
[hatched box] is a bigger open set containing γ and the inside of γ.

Let Ω = inside of γ.

Jordan curve theorem: A Jordan curve separates $\mathbb{C}$ into

$$\{\text{inside of } \gamma\} \cup \gamma \cup \{\text{outside of } \gamma\}.$$

Bounded — connected open sets — Unbounded



Green's Theorem:

$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

Pf of Cauchy's Thm: (C-R Eqns + Green's).

$$\int_\gamma f\,dz = \int_a^b f(z(t))\; z'(t)\,dt \qquad \begin{aligned} z(t) &= x(t) + i y(t) \\ f(x+iy) &= u + iv \end{aligned}$$

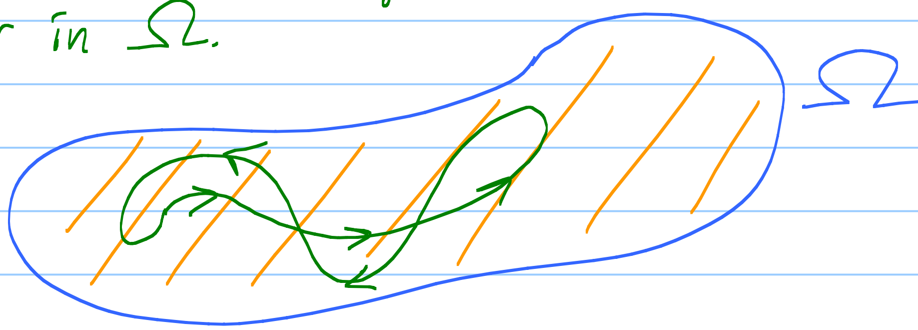
$$= \int_a^b \left[u(x(t),y(t)) + i\,v(x(t),y(t))\right]\left[x'(t) + i\,y'(t)\right] dt$$

$$= \int_a^b (u\dot{x} - v\dot{y})\,dt + i\int_a^b (u\dot{y} + v\dot{x})\,dt$$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx$$

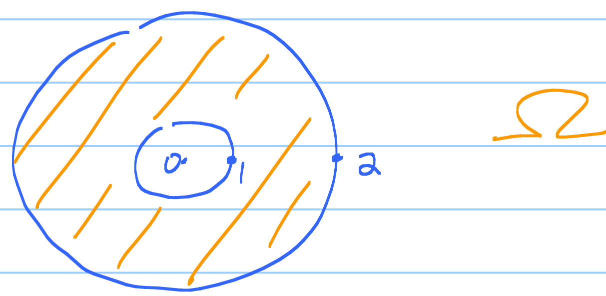
$$= \iint_{\Omega} \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\substack{=0 \\ \text{C-R Eqn \#2}}} dx\,dy + i \iint_{\Omega} \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{\substack{=0 \\ \text{C-R Eqn \#1}}} dx\,dy = 0 \checkmark$$

Cauchy Theorem: Suppose Ω is a domain with no holes (simply connected domain) and suppose f is analytic on Ω. Then $\int_\gamma f\,dz = 0$ for any closed contour in Ω.

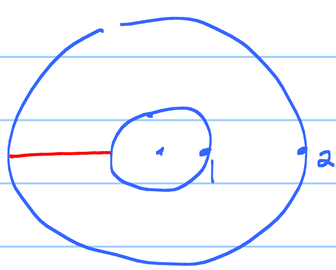


Use Cauchy Thm #1 on each little loop.

What's so bad about hole's?



EX 1: $\ln|z|$ is harmonic on Ω, but it does not have a harmonic conjugate on Ω.



$\tilde{\Omega} = \Omega - [-2,-1]$ has no holes

$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z$ is analytic on $\tilde{\Omega}$,

So $\mathrm{Arg}\, z$ is a harmonic conj for $\ln|z|$ on $\tilde{\Omega}$. Unique up to $+C$.

Aha! If $\ln|z| + iv$ were analytic on Ω, then v would be a harmonic conjugate on $\tilde{\Omega}$. So $v = \operatorname{Arg} z + C$.

Ouch! $\operatorname{Arg} z + C$ cannot be made continuous across $[-2,-1]$. No such v.

Ex 2: $\frac{1}{z}$ does not have an analytic antiderivative on annulus Ω.

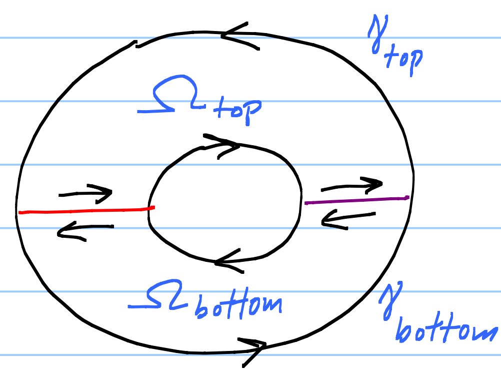
$$\int_{C_{3/2}} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\big(\underbrace{\tfrac{3}{2}e^{it}}_{z(t)}\big)} \big(\underbrace{i\tfrac{3}{2}e^{it}}_{z'(t)}\big)\,dt$$

$$= \int_0^{2\pi} i\,dt = 2\pi i \leftarrow \text{not zero!}$$

[If $\frac{1}{z} = F'(z)$, then $\int = 0$.]

Ex 3: Ex 2 shows that the obvious Cauchy Theorem for domains with holes fails.

Hmmm:



$$0 = \left(\int_{\gamma_{top}} + \int_{\gamma_{bottom}}\right) f\,dz = \int_{C_2} f\,dz + \int_{-C_1} f\,dz$$