Lesson 22 Radius of convergence

21, 22 due Wed.

Hadamard's Formula: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ has a R. of C. R;

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Series converges in $D_R(z_0)$, diverges in $\{z : |z-z_0| > R\}$.
Absolute and uniform convergence on $D_r(z_0)$ for r: $0 < r < R$.

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{N\to\infty} \sup_{n\geq N} \sqrt[n]{|a_n|}$$

Why do power series have a R of C?

Suppose $\sum_0^{\infty} a_n w^n$ converges. Then we know that $a_n w^n \to 0$ as $n\to\infty$. So $\exists N$ such that $|a_n w^n| < 1 \leftarrow \varepsilon = 1$ for $n > N$. Now suppose $|z| < |w|$. Then

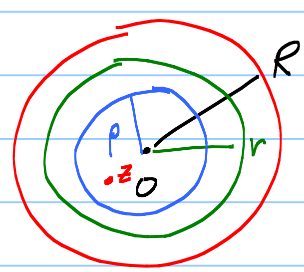
$$|a_n z^n| = \left|a_n w^n \frac{z^n}{w^n}\right| = |a_n w^n| \left|\frac{z}{w}\right|^n < 1 \underbrace{\left(\frac{|z|}{|w|}\right)}_{<1.}^n \text{ if } n > N,$$

Series converges on $D_r(0)$ where $r = |w|$.

Radius of conv $= \sup\{r : \sum_0^{\infty} a_n r^n \text{ converges}\}$

$= \sup\{r : |a_n| r^n \text{ is a bounded seq}\}$

Pf of Hadamard's:



$|z| < \rho < r < \underset{\substack{\uparrow \\ \text{Hadamard Formula}}}{R}$

Then $\dfrac{1}{R} < \dfrac{1}{r} < \dfrac{1}{\rho} < \dfrac{1}{|z|}$

$$\frac{1}{R} = \lim_{N\to\infty} \underbrace{\sup_{n\geq N} \sqrt[n]{|a_n|}}_{\text{non-increasing seq}}$$

$\sup_{n\geq N} \sqrt[n]{|a_n|}$ is sliding down to $\frac{1}{R}$. So eventually, it is $< \frac{1}{r}$.

$$\boxed{\sup_{n\geq N_0} \sqrt[n]{|a_n|} < \frac{1}{r}}$$

So $\sqrt[n]{|a_n|} < \frac{1}{r}$ for $n \geq N_0$.

Geom. Seq trick: n-th power: $|a_n| < \frac{1}{r^n}$

$$|a_n||z|^n < \frac{|z|^n}{r^n} < \frac{\rho^n}{r^n}$$

$$|a_n z^n| < \left(\underbrace{\frac{\rho}{r}}_{<1}\right)^n$$

So $\sum a_n z^n$ compares to conv geom series, and we get conv.

In fact, we get unif conv on $D_\rho(0)$:

$$\left| f(z) - \sum_{n=0}^{N-1} a_n z^n \right| = \left| \sum_{n=N}^{\infty} a_n z^n \right| \leq \sum_{n=N}^{\infty} |a_n z^n|$$

$$\underset{N>N_0}{\leq} \sum_{n=N}^{\infty} \left(\frac{\rho}{r}\right)^n = \left(\frac{\rho}{r}\right)^N \sum_{0}^{\infty} \left(\frac{\rho}{r}\right)^n$$

$$= \frac{\left(\frac{\rho}{r}\right)^N}{1-\frac{\rho}{r}} \to 0$$

So far: R of C $\geq R$.

Finally, show series diverges if $|z|>R$:

$$\frac{1}{|z|} < \underbrace{\frac{1}{R} < \sup_{n>N} \sqrt[n]{|a_n|}}$$

Aha! For each N, there is an n such that $\sqrt[n]{|a_n|} > \frac{1}{R}$.

$$\frac{1}{|z|} < \frac{1}{R} < \sqrt[n]{|a_n|}$$

$$\frac{1}{|z|^n} < \frac{1}{R^n} < |a_n| \leftarrow \text{mult by } |z^n|$$

$$1 < |a_n z^n|$$

For each N, $\exists\, n > N$ such that $|a_n z^n| > 1$. Terms don't $\to 0$ as $n \to \infty$. Divergence

<u>Prob</u>: Use power series to solve ODEs:

$$f'' - z f' - f = 0 \qquad \begin{cases} f(0) = 1 \\ f'(0) = 0 \end{cases}$$

Idea: Try $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $\qquad \begin{cases} a_0 = f(0) = 1 \\ a_1 = \frac{f'(0)}{1!} = 0 \end{cases}$

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$
$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$

$$f''(z) = 2a_2 + 3\cdot 2 a_3 z + 4\cdot 3 a_4 z^2 + \cdots$$
$$- \quad z f'(z) = a_1 z + 2a_2 z^2 + \cdots$$
$$- \quad f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

$$f'' - zf' - f = \underbrace{(2a_2 - a_0)}_{\text{must } = 0} + \underbrace{(3\cdot 2a_3 - 2a_1)}_{0} z + \underbrace{(4\cdot 3\, a_4 - 2a_2 - a_2)}_{0} z^2 + \cdots$$

n-th term: $\Big[\underbrace{(n+2)(n+1)a_{n+2} - n a_n - a_n}_{\text{must } = 0}\Big] z^n \leftarrow$ Recursion Formula:

$$a_{n+2} = \frac{(n+1)a_n}{(n+2)(n+1)} = \frac{a_n}{n+2}$$

$$\boxed{a_{n+2} = \frac{a_n}{n+2}}$$

$2a_2 - a_0 = 0 \qquad a_2 = \frac{1}{2}\underbrace{a_0}_{1} = \frac{1}{2}$

$3\cdot 2a_3 - 2a_1 = 0 \qquad a_3 = \frac{1}{3}\underbrace{a_1}_{0} = 0$

$4\cdot 3a_4 - 2a_2 - a_2 = 0 \qquad a_4 = \frac{a_2}{4} = \frac{1}{8}$

$\vdots$

etc.

Pattern: All odd terms $= 0$.

$$a_6 = \frac{1}{2^3 3!}$$
$$a_8 = \frac{1}{2^4 4!}$$
$$\vdots$$
$$a_{2n} = \frac{1}{2^n n!}$$

Why the factorial: $2\cdot 4\cdot 6\cdot 8\cdot 10\cdot \cdots (2n)$

$$(2\cdot 1)(2\cdot 2)(2\cdot 3)(2\cdot 4)\cdots(2n) = 2^n n!$$

Interesting point: Can determine R of C without actually computing a_n.

Trick: Use recursion formula and Ratio Test:

Ratio Test: $\displaystyle \lim \frac{|\text{Far out term}|}{|\text{term before}|} = \lim_{\substack{n\to\infty \\ n \text{ even}}} \left| \frac{a_{n+2} z^{n+2}}{a_n z^n} \right|$

Far out $\longrightarrow$ way far out

$$= \lim_{n\to\infty} \underbrace{\left|\frac{a_{n+2}}{a_n}\right|}_{\frac{1}{n+2}} |z|^2 = 0 < 1$$

Converges for all z. $R = \infty$.

What started out as wishful thinking is now completely valid!

Multiplying Power Series: $\displaystyle f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n$

$$(fg)(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{where } c\text{'s given by}$$

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

$$c_n = \sum_{k=0}^{n} a_k\, b_{n-k}$$

powers add up to n

Why: $c_n = \dfrac{(fg)^{(n)}(0)}{n!}$ ← use Leibnitz's Formula:

$$\frac{(fg)^{(n)}}{n!} = \sum_{k=0}^{n} \frac{f^{(k)}}{k!}\,\frac{g^{(n-k)}}{(n-k)!}$$

Leibnitz: $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}\, g^{(n-k)}$