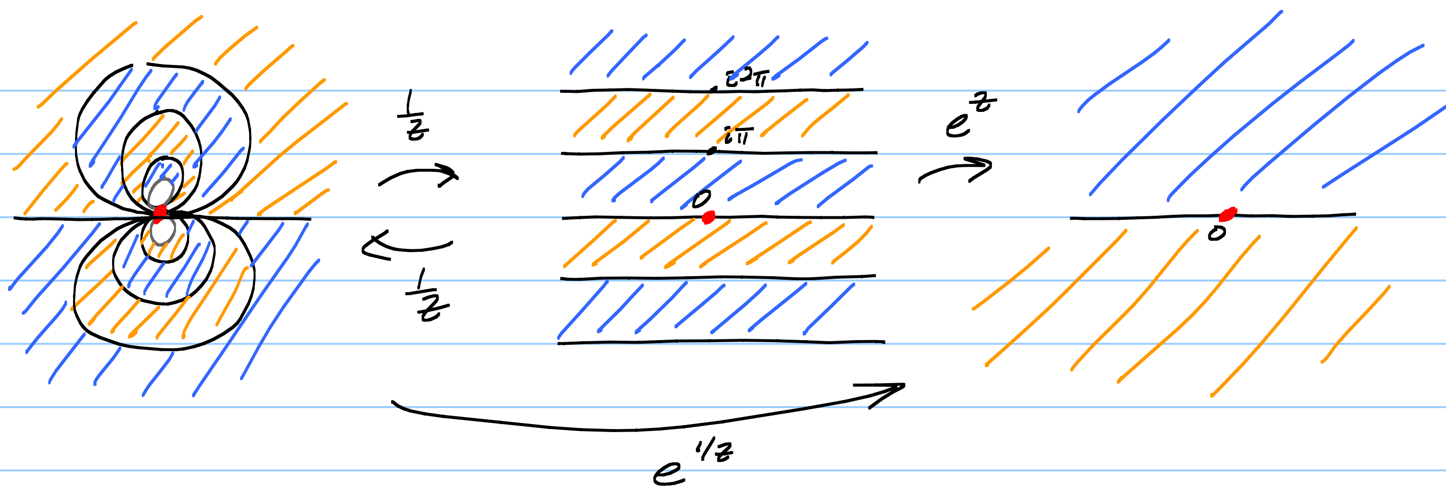
Lesson 25 Residue Theorem

23, 24, 25 due Wed.



$e^{1/z}$ shreds the complex plane near 0!

Three kinds of isolated singularities: f analytic on $D_\varepsilon(z_0) - \{z_0\}$

1) $\lim_{z \to z_0} f(z)$ exists $(\in \mathbb{C})$ $\Longleftrightarrow$ z_0 is removable.

no neg a_{-n}'s

2) $\lim_{z \to z_0} f(z) = \infty$ (meaning that, given $M > 0$, there is a δ with $0 < \delta < \varepsilon$ such that $|f(z)| > M$ when $z \in D_\delta(z_0) - \{z_0\}$) $\Longleftrightarrow$ f has a pole at z_0.

finitely many neg a_{-n}'s

3) (Picard Theorem) f hits every value in $\mathbb{C}$ or $\mathbb{C} - \{\text{one point}\}$ infinitely many times on $D_\delta(z_0) - \{z_0\}$ for every δ with $0 < \delta < \varepsilon$, $\Longleftrightarrow$ z_0 is an essential singuarity.

∞ many neg a_{-n}'s in Laurent expansion

EX: What kind of sing does $\cos(1/z)$ have at $z = 0$?

Hmmm $\cos\left(\frac{1}{t}\right)$ wiggles like crazy between $+1$ and -1 as $t \to 0+$

Rule out behavior (1). Not removable

Rule out (2) too. Not pole.

Must be essential.

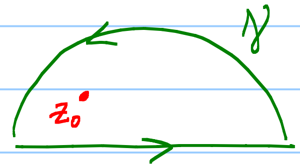
Most important a_n in Laurent expansion: $f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$n=-1$; $\quad a_{-1} = \frac{1}{2\pi i}\int_{C_r} f(z)\,dz \longleftarrow$ a_{-1} = Residue of f at z_0

$= \operatorname{Res}_{z_0} f$

Old HWK prob: $h(z) = \frac{A_N}{(z-z_0)^N} + \cdots + \frac{A_1}{(z-z_0)} + g(z)$ ← entire



$$\int_\gamma h\,dz = 2\pi i\, A_1 = 2\pi i \operatorname{Res}_{z_0} h$$

$\int_\gamma g\,dz = 0$ by Cauchy Thm.

$\int_\gamma \frac{A_n}{(z-z_0)^n}\,dz = A_n\int_\gamma \frac{d}{dz}\left[\frac{1}{-n+1}(z-z_0)^{-n+1}\right]dz = 0$ provided that $n\neq -1$

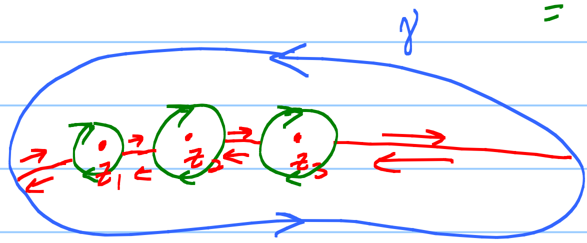
$\int_\gamma \frac{A_1}{z-z_0}\,dz = A_1\, 2\pi i \longleftarrow$ Cauchy Int formula : $f(z)=A_1$.

Residue Theorem: Suppose f is analytic inside and on a simple closed positively oriented (counterclockwise) curve γ except at finitely many isolated singularities $\{z_k\}_{k=1}^N$.

Then $\int_\gamma f\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}_{z_k} f$

= "$2\pi i$ times the sum of the residues of f inside γ"

Pf.:



$0 = \underbrace{\int_{Top} f\,dz}_{=0 \text{ by Cauchy}} + \underbrace{\int_{Bottom} f\,dz}_{=0}$

$$= \int_\gamma f\,dz + \sum_{k=1}^{N} \underbrace{\int_{-C_{\varepsilon_k}(z_k)} f\,dz}_{= -\int_{C_{\varepsilon_k}(z_k)} f\,dz \;=\; -2\pi i\, \mathrm{Res}_{z_k} f}$$

EX: $$e^{1/z} = 1 + \frac{1}{z} + \frac{1/2!}{z^2} + \frac{1/3!}{z^3} + \cdots$$

$\leftarrow \mathrm{Res}_0\, e^{1/z}$

$$\int_{C_1(0)} e^{1/z}\,dz = 2\pi i\, \mathrm{Res}_0\, e^{1/z} = 2\pi i$$

EX: $$\frac{\sin z}{z^4} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z^4} = \underbrace{\frac{1}{z^3} - \frac{1/3!}{z}}_{\text{Principal part}} + \underbrace{\frac{1}{5!}z + \cdots}_{\text{regular power series}}$$

residue term

$$\int_{C_1(0)} \frac{\sin z}{z^4}\,dz = 2\pi i\left(\frac{-1}{3!}\right) = -\frac{\pi i}{3}$$

EX: $$\frac{e^z}{z^2+1} = \frac{e^z}{(z-i)(z+i)} = \frac{1}{z-i}\left[\underbrace{\frac{e^z}{z+i}}_{\text{analytic near } i}\right]$$

$$= A_0 + A_1(z-i) + A_2(z-i)^2 + \cdots$$

$$= \frac{A_0}{z-i} + A_1 + A_2(z-i) + \cdots$$

$\leftarrow$ Res

Aha! $$\mathrm{Res}_i \frac{e^z}{z^2+1} = A_0 = \frac{f(i)}{0!} \quad \text{where } f(z) = \frac{e^z}{z+i}$$

$$= \frac{e^i}{i+i} = \frac{e^i}{2i}$$

Main tools: Algebra and Taylor formula.

Homework problems on Laurent expansions: Formula $a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ is nice but not useful in some cases.

EX: Find the Laurent exp of $f(z) = \frac{z}{(z+1)(z+2)}$ on the annulus $\{z : 1 < |z| < 2\}$.

One way: $f(z) = \frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$

partial fractions

Big hint: Geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots, \quad |r| < 1$

$1 < |z| < 2$ so $\frac{1}{|z|} < 1$ ← use for $\frac{1}{z+1}$

$\frac{|z|}{2} < 1$ ← use for $\frac{1}{z+2}$

$$\frac{1}{z+1} = \frac{1}{z} \cdot \frac{1}{1+\frac{1}{z}} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$r = -\frac{1}{z}$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots \checkmark$$

$$\frac{1}{z+2} = \frac{1}{2} \cdot \frac{1}{1+\frac{z}{2}}$$

$r = -\frac{z}{2}$

get regular power series.

Def^n: Suppose f analytic on $\{z : |z| > R\}$. Then we think of ∞ as being an isolated sing of f. The type of singularity of f at the point at infinity is the type of the singularity of $f(\frac{1}{z})$ at $z = 0$.

EX: e^z

$f(z) = e^z$

$f(1/z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1/2!}{z^2} + \cdots$ has an ess. sing at $z=0$.

So e^z has an ess. sing at ∞.