Lesson 27 More residue integrals

26, 27, 28 due Wed

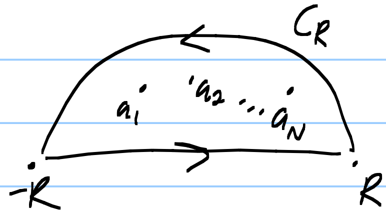
Fact: Suppose $P(z)$ and $Q(z)$ are polynomials and

1) $\deg P \le \deg Q - 2$
2) Q has no zeroes on $\mathbb{R}$.

Then
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{\substack{a_k \text{ zeroes} \\ \text{of } Q \text{ in UHP}}} \operatorname{Res}_{a_k} \frac{P}{Q}$$

$$\left(\text{Also} \quad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\,dx = 2\pi i \sum_{a_k} \operatorname{Res} \frac{P(z)}{Q(z)} e^{iz} \right)$$

Why:



Just need to show $\int_{C_R} \frac{P}{Q}\,dz \to 0$

$\left(\int_{C_R} \frac{P}{Q} e^{iz}\,dz \to 0 \text{ too} \right)$

Key: Basic Polynomial estimate: $P(x) = a_N z^N + \cdots + a_1 z + a_0$ $(a_N \neq 0)$. There is a radius $R > 0$ and constants $0 < A < B$ such that

numerator est
$$A|z|^N \le |P(z)| \le B|z|^N \quad \text{when } |z| > R_P$$

$$a|z|^M \le |Q(z)| \le b|z|^M \quad \text{when } |z| > r_Q$$

denom est

Now
$$\left| \int_{C_R} \frac{P}{Q}\,dz \right| \le \left(\operatorname{Max}_{C_R} \left|\frac{P}{Q}\right| \right) \pi R$$

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{B|z|^N}{a|z|^M} = \frac{B}{a}\frac{1}{R^{M-N}} \le \frac{B}{a}\frac{1}{R^2}$$

because $M - N \ge 2$

if $|z| = R > \operatorname{Max}(R_P, r_Q, 1)$

Easy to add e^{iz} in because $|e^{iz}| \le 1$ on UHP.

Trig Integrals EX $\displaystyle\int_0^{2\pi} \cos^6\theta \, d\theta = I$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{e^{i\theta} + \frac{1}{e^{i\theta}}}{2}$$

Hmmm, $z(\theta) = e^{i\theta}$ parametrizes $C_1(0)$, $0 \le \theta \le 2\pi$

Want a dz somehow. $\quad d\theta = \dfrac{\overbrace{z'(\theta)\, d\theta}^{dz}}{z'(\theta)}$

$\uparrow$ $ie^{i\theta} = i\,z(\theta)$

$$z'(\theta) = ie^{i\theta}$$

$$I = \int_0^{2\pi} \left[\frac{e^{i\theta} + \frac{1}{e^{i\theta}}}{2}\right]^6 \frac{\overbrace{ie^{i\theta}\, d\theta}^{dz}}{ie^{i\theta}}$$

$\uparrow$ $z = e^{i\theta}$

$$= \int_{C_1(0)} \left[\frac{z + \frac{1}{z}}{2}\right]^6 \frac{1}{iz}\, dz = 2\pi i \operatorname{Res}_0 f(z)$$

where $f(z) = \dfrac{1}{2^6 i}\left(z + \dfrac{1}{z}\right)^6 \dfrac{1}{z}$

residue term

$$= \frac{1}{2^6 i}\left(z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}\right)\cdot \frac{1}{z}$$

Ans: $I = 2\pi i \dfrac{20}{2^6 i} = \dfrac{2\pi\ 2\cdot 2\cdot 5}{2^6} = \dfrac{5\pi}{8}$

General method: $\displaystyle\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$, R rational

$$= \int_0^{2\pi} R\left(\frac{e^{i\theta} + \frac{1}{e^{i\theta}}}{2}, \frac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i}\right) \frac{1}{ie^{i\theta}} \underbrace{ie^{i\theta}\, d\theta}_{dz}$$

$\uparrow$ $z = e^{i\theta}$

$$= \int_{C_1(0)} \underbrace{R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{1}{iz}}_{f(z)}\, dz$$

$$= 2\pi i \sum_{a_k \in A} \operatorname{Res}_{a_k} f(z) \qquad \text{where } A = \text{poles } a_k \text{ of } f(z) \text{ inside } C_1(0)$$

[Need $R(\cos\theta, \sin\theta)$ not to blow up $0 \le \theta \le 2\pi$.]

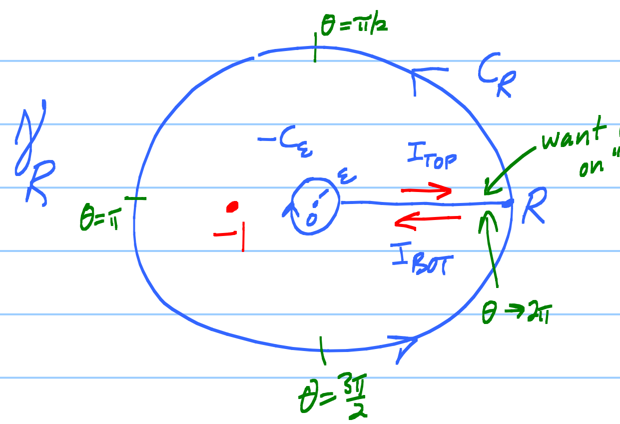
EX: $\displaystyle\int_0^\infty \frac{1}{\underbrace{\sqrt{x}\,(x+1)}}\, dx$ ← absolutely integrable

near zero $\sim \dfrac{1}{x^{1/2}}$ $\left(\displaystyle\int_0^\varepsilon \frac{1}{x^{1/2}}\, dx < \infty\right)$

$f(z) = \dfrac{1}{\sqrt{z}\,(z+1)}$

big $x > 0$ $\sim \dfrac{1}{x^{3/2}}$ $\left(\displaystyle\int_R^\infty \frac{1}{x^{3/2}}\, dx < \infty\right)$

Branch of $\sqrt{z} = e^{\frac{1}{2}\log z}$ for a branch of $\log z = \operatorname{Ln}|z| + i\theta$, $\theta \in \arg z$



I_{Top} : $z(t) = t \quad \varepsilon \le t \le R$

$$\int_{I_{Top}} \frac{1}{\sqrt{z}\,(z+1)}\, dz = \int_\varepsilon^R \frac{1}{\sqrt{t}\,(t+1)}\, dt$$

$$\int_{I_{BOT}} \frac{1}{e^{\frac{1}{2}\log z}(z+1)}\, dz = \int_{I_{BOT}} \frac{1}{e^{\frac{1}{2}(\operatorname{Ln}|z| + i2\pi)}(z+1)}\, dz$$

$$= -\int_{I_{Top}} \frac{1}{\underbrace{e^{i\pi}}_{=-1} e^{\frac{1}{2}\operatorname{Ln}|z|}(z+1)}\, dz \qquad z = t,\ \varepsilon \le t \le R$$

$$= \int_\varepsilon^R \frac{1}{\sqrt{t}\,(t+1)}\, dt \quad \leftarrow \text{Want!}$$

Nice fact: $\left|\sqrt{z}\right| = \left|e^{\frac{1}{2}\log z}\right| = \left|e^{\frac{1}{2}(\ln|z| + i\theta)}\right|$

$$= e^{\frac{1}{2}\ln|z|}\underbrace{\left|e^{\frac{1}{2}i\theta}\right|}_{=1} = \sqrt{|z|}$$

[Or $|\sqrt{z}||\sqrt{z}| = |\sqrt{z}^2| = |z|$.]

Also $|z^\alpha| = |z|^\alpha$ when $\alpha \in \mathbb{R}$.

Next $\left|\int_{C_R} f(z)\,dz\right| \le \left(\max_{C_R}|f|\right)2\pi R \le \frac{1}{\sqrt{R}(R-1)}2\pi R \to 0$ as $R\to\infty$

if $R>1$

$$\left|\int_{-C_\varepsilon} f\,dz\right| \le \left(\max_{C_\varepsilon}|f|\right)2\pi\varepsilon \le \frac{1}{\sqrt{\varepsilon}(1-\varepsilon)}2\pi\varepsilon \to 0 \text{ as } \varepsilon\to 0$$

Hmm: $|z+1| = |z-(-1)| \ge \left||z|-|-1|\right| = |\varepsilon-1| = 1-\varepsilon$! $\varepsilon<1$

Last step: $\operatorname{Res}_{-1}\frac{1}{\sqrt{z}(z-(-1))} = \frac{1}{\sqrt{-1}} = \frac{1}{e^{\frac{1}{2}(\ln|-1|+i\pi)}} = \frac{1}{e^{i\pi/2}} = \frac{1}{i} = -i$

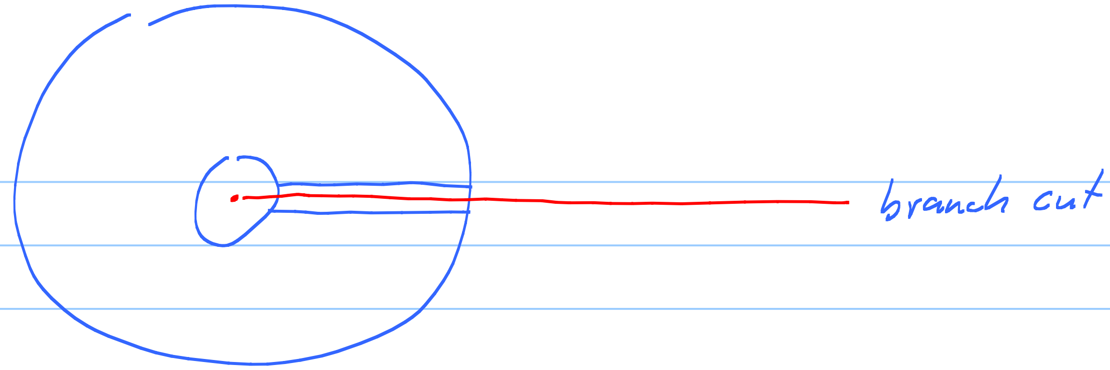
[$\operatorname{Res}_a \frac{f(z)}{z-a} = f(a)$ when f analytic near a]

and $\int_{\gamma_R} f\,dz = 2\pi i \operatorname{Res}_{-1} f$

$$\underbrace{\int_{C_R}}_{\to 0} + \int_{I_{TOP}} + \underbrace{\int_{-C_\varepsilon}}_{\to 0} + \int_{I_{BOT}} = 2\pi i(-i) = 2\pi$$

$= 2\int_\varepsilon^R \frac{1}{\sqrt{x}(x+1)}\,dx$

First let $\varepsilon\to 0$. Then $R\to\infty$. Get $I = \pi$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