Lesson 31 Harmonic functions composed with analytic functions are harmonic too (7.1)

29, 30 due Mon.
Exam 2 in class on Wed. Nov. 8

Fact: u is harmonic on a domain if and only if it is locally the real part of an analytic function, i.e., for each z_0 in the domain Ω, there is a disc $D_r(z_0) \subset \Omega$ and an analytic fcn f on $D_r(z_0)$ such that $u = \operatorname{Re} f$ on $D_r(z_0)$.

Note: If Ω is simply connected, can $u = \operatorname{Re} f$ with f analytic on Ω.

EX: $\operatorname{Ln}|z|$ on $\{z : 1 < |z| < 2\}$ has no harm conj on annulus.

Thm:

$u = \operatorname{Re} G$ on $D_\varepsilon(w_0)$
G analytic

Then $u \circ f$ is harmonic on Ω_1.

Why: Near z_0 $\quad u(f(z)) = \operatorname{Re} \underbrace{G \circ f}_{\text{analytic near } z_0}$

The hard way: $\Delta(u \circ f) = \cdots$ chain rule plus C-R Eqns $\cdots \equiv 0$

Review: Formulas for residues. f, g analytic

1) $$\operatorname{Res}_{z_0} \frac{f(z)}{z - z_0} = f(z_0) \qquad \frac{a_0 + a_1(z - z_0) + \cdots}{z - z_0} = \frac{a_0}{z - z_0} + \text{power series}, \quad a_0 = f(z_0)$$

Remark: Cauchy Integral Formula

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz = f(z_0) = \frac{1}{2\pi i}\left[2\pi i \operatorname{Res}_{z_0} \frac{f(z)}{z - z_0} \right] \checkmark$$

2) If g has simple zero at z_0, then

$$\operatorname{Res}_{z_0} \frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$$

<u>EX</u>: $$\operatorname{Res}_{\underbrace{e^{i\pi/4}}_{\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}}} \frac{e^{2z}}{z^4+1} \quad \begin{array}{l} \leftarrow f(z) = e^{2z} \\ \leftarrow g(z) = z^4+1 \qquad g(e^{i\pi/4}) = 0 \\ g'(z) = 4z^3 \qquad g'(e^{i\pi/4}) = 4\underset{\substack{\uparrow \\ \text{not zero}}}{(e^{i\pi/4})^3} \end{array}$$

$$\operatorname{Res} = \frac{e^{(\sqrt{2}+i\sqrt{2})}}{4\, e^{i3\pi/4}}$$

I hate formula in box on p. 310:

<u>Thm 1</u>: If f has a pole of order m at z_0, then

$$\operatorname{Res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

True, but better to manipulate power series.

<u>EX</u>: Suppose f, g analytic and g has a <u>double</u> zero at z_0.

$$\operatorname{Res}_{z_0} \frac{f}{g} = ?$$

$$\underset{\substack{\uparrow \\ a_0}}{g(z_0) = 0}, \quad \underset{\substack{\uparrow \\ a_1}}{g'(z_0) = 0}, \quad \text{but} \quad g''(z_0) \neq 0.$$

$$g(z) = \underset{\substack{\uparrow \\ a_2 = \frac{g''(z_0)}{2!}}}{a_2}(z-z_0)^2 + \underset{\substack{\uparrow \\ a_3 = \frac{g'''(z_0)}{3!}}}{a_3}(z-z_0)^3 + \cdots = (z-z_0)^2 \Big[\underbrace{a_2 + a_3(z-z_0) + \cdots}_{G(z) \text{ analytic}} \Big]$$

Notice that
$$\boxed{\begin{array}{l} G(z_0) = a_2 \neq 0 \\ G'(z_0) = a_3 \end{array}}$$

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-z_0)^2\, G(z)} = \frac{1}{(z-z_0)^2} \Bigg[\underbrace{\overbrace{f(z)/G(z)}^{H(z)}}_{A_0 + A_1(z-z_0) + \cdots} \Bigg] \leftarrow \text{analytic near } z_0$$

$$= \frac{A_0}{(z-z_0)^2} + \frac{A_1}{(z-z_0)} + \underbrace{A_2 + \cdots}_{\text{power series}}$$

Ans: $\text{Res}_{z_0} \dfrac{f}{g} = A_1 = \dfrac{H'(z_0)}{1!} = \dfrac{d}{dz}\left[\dfrac{f}{G}\right]\Bigg|_{z_0}$

$$= \frac{f'(z_0)\overbrace{G(z_0)}^{a_2} - \overbrace{G'(z_0)}^{a_3} f(z_0)}{G(z_0)^2}$$

$$= \frac{f'(z_0)\cdot \frac{1}{2} g''(z_0) - \frac{1}{6} g'''(z_0) f(z_0)}{\left[\frac{1}{2} g''(z_0)\right]^2}$$

EX: $\text{Res}_i \dfrac{\text{Log}\, z}{(z^2+1)^2}$ ← don't memorize or use

$$\frac{\text{Log}\, z}{(z-i)^2 (z+i)^2} = \frac{1}{(z-i)^2}\left[\frac{\overbrace{\text{Log}\, z}^{H(z)}}{(z+i)^2}\right]$$

← analytic near i

$A_0 + A_1(z-i) + \cdots$

$$= \frac{A_0}{(z-i)^2} + \frac{A_1}{z-i} + \underbrace{A_2 + \cdots}_{\text{power series}}$$

$$\text{Res}_i = A_1 = \frac{H'(i)}{1!} = \frac{\frac{d}{dz}\left[\frac{\text{Log}\, z}{(z+i)^2}\right]}{1!}\Bigg|_{z=i}$$

$$= \frac{\left(\frac{1}{z}\right)(z+i)^2 - 2(z+i)\,\text{Log}\, z}{\left[(z+i)^2\right]^4}\Bigg|_{z=i}$$

$\text{Log}\, i = \text{Ln}|i| + i\,\text{Arg}\, i$
$= 0 + i\frac{\pi}{2}$

Prob: Suppose f is entire and f misses $[-1,1]$. Show f must be constant.

Hint: f entire. Then $f(1/z)$ has an isolated singularity at $z=0$!

If sing is essential, then f plasters all of $\mathbb{C}$, with perhaps one exception. $\lightning$ Not ess.

If sing removable, then $\lim_{z\to 0} f(\frac{1}{z}) = a_0$.

Then $\lim_{z\to\infty} f(z) = a_0$. Aha! That makes f bounded entire.

So $f \equiv c = a_0$

If sing is a pole, then $\lim_{z\to 0} f(\frac{1}{z}) = \infty$.

f has a pole at infinity

Claim: f must be a poly.

Fact: An entire fcn with a pole at ∞ must be a poly.

Fund Thm Alg $\Rightarrow$ polys don't miss any values in $\mathbb{C}$.
So $f \neq$ poly.

Pf of claim: $f(z) = \sum_0^\infty a_n z^n$

$$f(\tfrac{1}{z}) = \sum_0^\infty a_n \frac{1}{z^n} \leftarrow \text{pole!} \quad \Rightarrow \quad \sum_0^N a_n \frac{1}{z^n}$$

Aha! $f(z) = \sum_0^N a_n z^n$ = poly.

Jukovsky map: