Lesson 32 Conformal mapping

Prob 2 on Exam 2: f analytic on $\mathbb{C}-\{a\}$ with a simple pole at a and a removable singularity at ∞. Then $f(z)=\frac{A}{z-a}+B$.

Why: a simple means: $f(z)=\underbrace{\frac{A}{z-a}}_{\text{Princ. Part}}+$ power series in $(z-a)$ is L. Exp. of f near a, where $A=\text{Res}_a f$.

Step 1: $f(z)-\frac{A}{z-a}$ has a removable sing at $z=a$.

Let $F(z)=f(z)-\frac{A}{z-a}$ with the proper value at a to make it entire.

Step 2: ∞ removable for f means $\lim_{z\to\infty} f(z)$ exists. Say $=B$.

Aha! $$\lim_{z\to\infty}F(z)=\lim_{z\to\infty}f(z)-\lim_{z\to\infty}\frac{A}{z-a}$$
$$=B-0=B$$

So F is a bounded entire fcn. Liouville's $\Rightarrow F\equiv$ const.

Since $\lim_{z\to\infty}F(z)=B$, the const must be B.

$$F(z)=f(z)-\frac{A}{z-a}\equiv B. \quad \checkmark$$

Partial Fractions: Suppose $P(z)$ and $Q(z)$ are polynomials with no common factors and $\deg P<\deg Q$. Let the zeroes of Q be $a_1,a_2,\dots,a_N$ and suppose the mult. of a_k is m_k.

Then $$\frac{P(z)}{Q(z)}=\sum_{n=1}^{N}\underbrace{\sum_{k=1}^{m_k}\frac{A_{nk}}{(z-a_n)^k}}_{\text{Princ. Part of } \frac{P}{Q} \text{ at } a_n = R_n}$$

Why: $\frac{P}{Q}$ has a pole of order m_n at a_n. Let R_n denote the princ part at a_n.

Claim: $F(z) = \frac{P}{Q} - \sum_{n=1}^{N} R_n$ has removable sing at each a_n.

Why: Near a_n: $F(z) = \underbrace{\left(\frac{P}{Q} - R_n\right)}_{\text{Removable sing at } a_n} - \underbrace{\sum_{k \neq n} R_k}_{\text{analytic near } a_n}$

Give F the proper values at each a_n to make it entire.

Basic Poly est: $A|z|^{\deg P} \le |P(z)| \le B|z|^{\deg P}$ for $|z| > R_P$

$a|z|^{\deg Q} \le |Q(z)| \le b|z|^{\deg Q}$ for $|z| > R_Q$

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{B}{a} \frac{1}{|z|^{\underbrace{\deg Q - \deg P}_{>0}}} \quad \text{when } |z| > \operatorname{Max}(R_P, R_Q).$$

$\longrightarrow 0$ as $|z| \to \infty$.

Also: $\frac{1}{(z-a_n)^k} \to 0$ as $|z| \to \infty$ too.

So $\lim_{z\to\infty} F(z) = \underbrace{\lim_{z\to\infty} \frac{P}{Q}}_{0} - \sum_{n=1}^{N} \underbrace{\lim_{z\to\infty} R_n}_{0} = 0$

Liouville's $\Rightarrow$ $F \equiv$ const. $F(z) \to 0$ as $z \to \infty$ implies the const $= 0$. So $\frac{P}{Q} = \sum_{n=1}^{N} R_n$ ✓

Freshman Calculus fact: P, Q have real coeff and $(x^2 + bx + c)$ is a factor of Q with complex roots $w, \bar{w}$

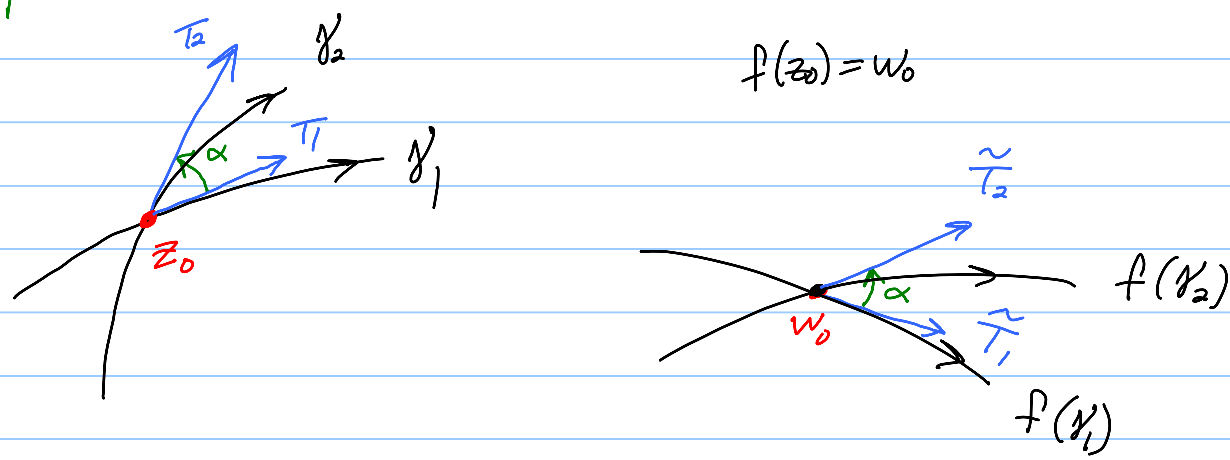
Partial Fractions : $\dfrac{A}{x-w} + \dfrac{B}{x-\bar{w}} = \dfrac{\alpha x+\beta}{x^2+bx+c}$

See $B = \bar{A}$

Today: Lesson 32

Suppose $f(z)$ is analytic near z_0 and $f'(z_0) \neq 0$.

Then f is conformal at z_0, meaning that angles are preserved there.



$\gamma_j : z_j(t)$, $z_j(0) = z_0$, $z_j'(0) \neq 0$

$T_j = z_j'(0) = x_j'(0) + i y_j'(0)$

$f(\gamma_j) : f(z_j(t))$

$\tilde{T}_j = f'(\underbrace{z_j(0)}_{z_0}) \underbrace{z_j'(0)}_{T_j}$

Aha! Chain rule #2 : $\dfrac{d}{dt} f(z(t)) = f'(z(t))\, z'(t)$

complex f' $\qquad$ $\mathbb{R} \to \mathbb{C}$ deriv.

$$\boxed{\tilde{T}_j = f'(z_0)\, T_j}$$

Suppose $f'(z_0) = \underbrace{Re^{i\theta}}$

Stretch by R and rotate by θ.

So angle α is preserved (in sense and size).

Big fact : Reverse is true (with some minor hitches).

Suppose $(x,y) \mapsto (u(x,y), v(x,y))$ is $\mathcal{C}^1$-smooth and the Jacobian is non-zero at (x_0, y_0).

$$\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

If map preserves angles at (x_0, y_0), then u, v satisfy the C-R Eqns.

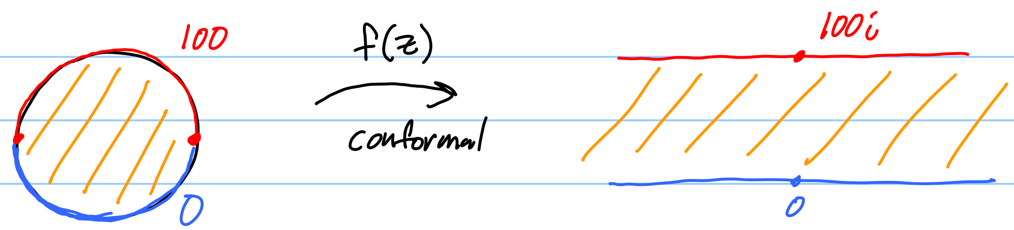
Why: Sophomore calculus, analytic geom.

Jacobian matrix: $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ preserves angles

exactly when it has the form $R \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Aha! See C-R Eqns.

Exciting news:



Sol^n

$V = y = \operatorname{Im} z$

Heat problem (steady state)

Sol^n: $\operatorname{Im}(V \circ f)$

Riemann Mapping Theorem: If $\Omega \neq \mathbb{C}$ is a simply connected domain, then there is 1-1 onto conformal mapping f of Ω onto $D_1(0)$.

Words: Me: Möbius Transformations: $\dfrac{z-a}{1-\bar{a}z}$, $|a|<1$.

Linear Fractional Transformation: $\dfrac{az+b}{cz+d}$, $ad-bc\neq 0$

LFTs

Book calls these Möbius Transf.